Computing Generalized Racah and Clebsch-Gordan Coefficients for U(N) groups

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May 9, 2006

Abstract

After careful introduction and discussion of the concepts involved, procedures are developed to compute Racah and Clebsch-Gordan coefficients for general r-fold tensor products of the $U(N)$ groups. In the process, the multiplicity of a given irreducible representation (irrep) in the direct sum basis is computed, and generalized Casimir operators are introduced to uniquely label the multiple irreps. Procedures to compute the Gel'fand-Zetlin basis are also included, as the Clebsch-Gordan coefficients are computed in this basis. Examples from $SU(2)$, $SU(3)$, and $SU(4)$ are provided.

1 Introduction and Background

In quantum mechanics, one describes systems as being in certain states. The observable quantities of the system are then related to underlying symmetry groups. For example, given the state of a hydrogen atom, an observable quantity would be the value of the orbital angular momentum of the electron about the proton. Angular momentum is a physical realization of the symmetry group of rotational transformations, which can be related to the group $SO(3)$, the group of special orthogonal matrices over three dimensions, or the group $SU(2)$, the special unitary group over two dimensions.

Many other physical observables come from the unitary and special unitary symmetry groups. In particular, all types of orbital and intrinsic angular momentum are related to $SU(2)$. Isospin also comes from $SU(2)$, being a subset of the $SU(3)$ flavor groups relating the various types of quarks: up, down, etc. In addition, quark color belongs with an $SU(3)$ group, and even higher $SU(N)$ groups arise in some circumstances.

Often one needs to couple various representations together. For example, in the hydrogen atom, the electron has an intrinsic angular momentum (called spin) and an orbital angular momentum. Thus to determine the total angular momentum of the electron, one must combine the orbital angular momentum and the intrinsic angular momentum. Each observable value of spin and orbital angular momentum correspond to elements of irreducible representations (irreps) of $SU(2)$. Mathematically, this combining of spin and angular momentum is the tensor product of representations of a group. Tensor products of irreps are not generally irreducible, and thus can be written as a direct sum of their

irreducible portions. One can look at the tensor product from the tensor product basis (as a product of irreps) or can view the tensor product from the direct sum basis (as the sum of its irreducible portions). Moving between the product basis and the direct sum basis is the mathematical equivalent of combining the constituent values to determine the total value.

The process is equivalent for any quantities obeying symmetry groups, and can easily be generalized. Consider a physical system of r-number of quantities, each an irrep of a $U(N)$ or $SU(N)$ group. Since for any N, $SU(N)$ is a subgroup of $U(N)$, we need only consider $U(N)$ groups at present. The $SU(N)$ groups are trivially related, as shall be later shown. The physical question of what possible total values of the system can result from the r given values being combined, is analogous to the question of what is the direct sum basis of a given r-fold tensor product and what is the relation between elements of the tensor product basis and elements of the direct sum basis.

This question is highly non-trivial, as it is further complicated by the possibility of a given irrep occurring multiple times in the direct sum. The Casimir operators labeling the irreps are then degenerate over this multiplicity, and thus provide no well defined basis or basis labels among the space where the multiplicity occurs. We can restrict our attention to one particular irrep at a time in the tensor product basis. Yet before one can consider either Racah coefficients or Clebsch-Gordan coefficients, one must determine the multiplicity of the irrep in question, i.e. number of times it occurs in the direct sum. This itself is, in general, quite difficult [3]. One must also determine a basis and labels for the multiplicity: that is, one must separate and label each multiple occurrence. However, traditional Casimir operators are unable to do this.

This paper shows how to realize concrete representations of the irreps and the multiplicity. Generalized Casimir operators will be introduced that both determine and label bases related to coupling schemes of the groups in the tensor product. Thus once the multiplicity is determined and given a labeled basis, Racah and Clebsch-Gordan coefficients can finally be computed. Racah coefficients are defined as the relation between coupling schemes, i.e. the change between bases in the multiplicity labels. Clebsch-Gordan coefficients connect the direct sum basis with elements of the tensor product basis and visa versa. However, one challenge yet remains, as in the past there is has been no method for computing concrete basis elements within any $U(N)$ irrep for $N > 2$. This paper gives procedures to compute a concrete realization the Gel'fand-Zetlin basis, as well as methods for computing Clebsch-Gordan coefficients in this basis.

2 Particular Case: $SU(2)$

2.1 Clebsch-Gordan Coefficients

In $SU(2)$, states are usually labeled in Dirac notation by $|j, m\rangle$. Since angular momentum is a vector operator, j is considered the magnitude of the angular momentum operator, and m the component of the angular momentum in one particular direction, usually chosen to be the z-direction. Mathematically, j is related to the eigenvalue of a Casimir Operator and labels the irrep, while m is the eigenvalue of an operator from the Lie algebra, usually chosen to be

J_z corresponding with the above choice of direction. These form a maximally commuting set, and thus fully label the states. One could equivalently consider the vector space of the irrep. The j value would specify which space, and each m value would refer to a specific basis element in that space.

For $U(N)$ groups of larger N, we will need additional Casimir operators to label the irrep and will need more than one Lie algebra element to form a maximally commuting set to label the states. In the general Gel'fand-Zetlin basis, each element is labeled by an entire tableau of eigenvalues. This basis will be discussed in more detail in Section 3.3.

Consider then the product of two $SU(2)$ states. An element of the tensor product basis in Dirac notation is: $|j_1, m_1\rangle |j_2, m_2\rangle$. A basis element of the direct sum basis can be specified as $|J, M; j_1, j_2\rangle$, with J being the irrep of the total angular momentum, M being the component of the total angular momentum, and j_1, j_2 denoting the irreps of the product. Since both the tensor product basis and direct sum basis span the same space, there exists a change of basis relating the two. The matrix elements of this change of basis are precisely the Clebsch-Gordan coefficients. Specifically, a Clebsch-Gordan coefficient is the overlap between a basis element in the direct sum basis with an element of the tensor product basis. In Dirac notation, these coefficients are denoted $\langle J, M; j_1, j_2 | j_1, m_1\rangle |j_2, m_2\rangle$. Notational conventions vary, although one convention is as follows:

$$\langle J, M; j_1, j_2 | j_1, m_1\rangle |j_2, m_2\rangle = C^{J, j_1, j_2}_{M, m_1, m_2}. \tag{1}$$

Using the above notation, one can construct the direct sum basis in terms of the tensor product basis as follows:

$$\begin{aligned} |J, M; j_1, j_2\rangle &= \sum_{m_1, m_2} |j_1, m_1\rangle |j_2, m_2\rangle \langle j_1, m_1| \langle j_2, m_2 | J, M; j_1, j_2\rangle \\ &= \sum_{m_1, m_2} C^{J, j_1, j_2}_{M, m_1, m_2} |j_1, m_1\rangle |j_2, m_2\rangle . \end{aligned} \tag{2}$$

For example, if $j_1 = j_2 = \frac{1}{2}$ the Clebsch-Gordan coefficients give the following change of basis:

$$\begin{aligned} \left|1, 1; \frac{1}{2}, \frac{1}{2}\right\rangle &= \left|\frac{1}{2}, \frac{1}{2}\right\rangle \left|\frac{1}{2}, \frac{1}{2}\right\rangle , \\ \left|1, 0; \frac{1}{2}, \frac{1}{2}\right\rangle &= \frac{1}{\sqrt{2}} \left(\left|\frac{1}{2}, -\frac{1}{2}\right\rangle \left|\frac{1}{2}, \frac{1}{2}\right\rangle + \left|\frac{1}{2}, \frac{1}{2}\right\rangle \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \right) , \\ \left|1, -1; \frac{1}{2}, \frac{1}{2}\right\rangle &= \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \left|\frac{1}{2}, -\frac{1}{2}\right\rangle , \\ \left|0, 0; \frac{1}{2}, \frac{1}{2}\right\rangle &= \frac{1}{\sqrt{2}} \left(\left|\frac{1}{2}, -\frac{1}{2}\right\rangle \left|\frac{1}{2}, \frac{1}{2}\right\rangle - \left|\frac{1}{2}, \frac{1}{2}\right\rangle \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \right) . \end{aligned} \tag{3}$$

Clebsch-Gordan coefficients for r-fold tensor products could be defined as

$$\langle J, M; j_1, j_2, \ldots j_r | j_1, m_1\rangle |j_2, m_2\rangle \ldots |j_r, m_r\rangle . \tag{4}$$

However, such definition does not take into account multiplicity of irreps. Furthermore, these coefficients are very difficult to compute even for small values of

r, and such formulas are not even known for larger values of r. The procedures outlined in this paper do not determine a closed form expression, but rather allow for the direct computation for any given tensor product.

To generalize to larger $U(N)$ groups, one needs replace j with (several) values labeling the irrep, and replace m with (several) values specifying a basis element. Since the irreps are always labeled by eigenvalues of Casimir operators, it is clear how to replace the j values. However, in larger $U(N)$ groups there is no longer a unique basis, and thus it is unclear what m generalizes to in Equation (4). The Gel'fand-Zetlin basis is a preferable choice for the basis, but until now there has been no general method of computing the basis. This paper outlines how to compute such a basis, and later Equation (4) will be properly generalized to hold for any $U(N)$ group (see Equation (30)).

2.2 Racah Coefficients

For $SU(2)$, the rules for computing what elements are in the direct sum from a given tensor product are fairly simple. The m values simply add, and thus $M = m_1 + m_2$. The J values are $j_1 + j_2$, $j_1 + j_2 - 1$, ..., $|j_1 - j_2|$. More precisely,

$$j_1 \otimes j_2 = (j_1 + j_2) \otimes (j_1 + j_2 - 1) \otimes \ldots \otimes (|j_1 - j_2| + 1) \otimes |j_1 - j_2|. \quad (5)$$

Thus for $j_1 = j_2 = \frac{1}{2}$, J can equal 1 or 0 since $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$.

Alternately, consider the three fold tensor product $j_1 = j_2 = j_3 = \frac{1}{2}$. Showing each step explicitly, and leaving off the kets for brevity, we have

$$\begin{aligned} \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} &= \left(\frac{1}{2} \otimes \frac{1}{2}\right) \otimes \frac{1}{2} \\ &= (1 \oplus 0) \otimes \frac{1}{2} \\ &= \left(1 \otimes \frac{1}{2}\right) \oplus \left(0 \otimes \frac{1}{2}\right) \\ &= \left(\frac{3}{2} \oplus \frac{1}{2}\right) \oplus \frac{1}{2} \\ &= \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}. \end{aligned} \quad (6)$$

Note the irrep $\frac{1}{2}$ occurs twice, or with multiplicity two, in the direct sum. No other Casimir operators from the Lie algebras of the irreps can distinguish the two $\frac{1}{2}$ irreps. However, one can consider the intermediate steps: one is the result of $1 \otimes \frac{1}{2}$, while the other was the result of $0 \otimes \frac{1}{2}$. We could then distinguish the two by the intermediate value, as say $|J, j_{int}\rangle_{1,2}$, where j_{int} is the intermediate value of $j_1 \otimes j_2$. Thus the degeneracy in this case is broken and Equation (6) yields

$$\left|\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}\right\rangle = \left|\frac{3}{2}, 1\right\rangle_{1,2} \oplus \left|\frac{1}{2}, 1\right\rangle_{1,2} \oplus \left|\frac{1}{2}, 0\right\rangle_{1,2}. \quad (7)$$

Likewise, one could couple any other two first, say j_2 with j_3 or j_1 with j_3. In this case, the states could be labeled $|J, j_{int}\rangle_{2,3}$ or $|J, j_{int}\rangle_{1,3}$. Likewise, in an r-fold tensor product, a basis can be given by the $r - 2$ intermediate values corresponding to a particular coupling scheme. Racah coefficients are defined as the matrix elements relating the bases given by different coupling schemes. In

particular, $SU(2)$ Racah coefficients for 3-fold tensors products could be written as

$$\langle J, (j)^{(A)}_{int} | J, (j)^{(B)}_{int} \rangle \,, \tag{8}$$

where $(j)^{(A)}_{int}$ labels a particular set of intermediate values in some coupling scheme A and $(j)^{(B)}_{int}$ labels a particular set of intermediate values in a second coupling scheme B.

The Wigner 6-, 9-, and 15-j symbols are (up to a phase) specific example of Racah coefficients. (For example, see [9]) However, these Wigner symbols are only defined for certain coupling schemes of 3, 4 and 5-fold tensor products. To compute more general coupling schemes is quite difficult, when even possible. Usually, Racah coefficients are computed as sums of Clebsch-Gordan coefficients. However, Clebsch-Gordan coefficients are themselves complicated to compute and are basis dependent (for $SU(2)$, this means they depend on the m value), whereas Racah coefficients are basis independent. One should avoid the difficulties of determining the basis when computing basis independent quantities, and thus should compute Racah coefficients directly.

In addition, the method of using intermediate values to break the degeneracy does not directly generalize to all $U(N)$ groups. For example, the 2-fold product of the eight dimensional irrep of $SU(3)$ is degenerate. Yet, since this is a two fold product, there are no apparent intermediate values to use. Later, generalized Casimir operators will be introduced to address these issues. These operators can be shown to be a direct generalization of the above intermediate value method, whose eigenvalues correspond to the intermediate coupling values when such exist.

The procedures outlined in this paper compute Racah and Clebsch-Gordan coefficients independently. The basis independence of the final Racah coefficient computation will be noted in Section 3.2. In the process generalized Casimir operators specify the coupling scheme and label the basis states corresponding to that coupling scheme. Thus the procedures allow the computation of Racah coefficients between any choice of coupling schemes for any general r-fold tensor product.

3 Theoretical Procedures

3.1 The Representation Space

The beginning of this section follows very closely the work published by the author and his advisors. [2] To facilitate the computation of the coefficients, we choose the concrete representation of polynomials over complex variables. Let $(m) := (m_1, m_2, \ldots m_N)$ denote an N-tuple of integers with the so-called dominant condition:

$$m_1 \geq m_2 \geq \ldots \geq m_N \geq 0. \tag{9}$$

Consider next the vector space defined with the left covariant condition, as follows:

$$V^{(m)} = (f : \mathbb{C}^{N \times N} \to \mathbb{C} \mid f(bz) = \pi^{(m)}(b) f(z)), \tag{10}$$

where f is a polynomial in $N \times N$ complex variables (denoted by z) and b is an element of the subgroup B of lower triangular matrices. $\pi(b)$ is a (one-dimensional) representation of B, defined by $\pi^{(m)} = b_{11}^{m_1} \ldots b_{NN}^{m_N}$. This repre-

sentation space is a subset of an infinite dimensional Bargman-Seigel-Fock space [5].

A representation of $U(N)$ on $V^{(m)}$ is given by $\left(R_g^{(m)} f\right)(z) = f(zg)$. According to the Borel-Weil theory [8], this representation is irreducible since (m) satisfies the so-called dominant condition mentioned above. The inner product on $V^{(m)}$ can be given as a differentiation inner product as follows:

$$(f, f) = f(D)\overline{f}(\overline{z})\big|_{z=0}, \tag{11}$$

where $f(D)$ denotes the differential operator obtained by replacing $z_{i\alpha}$ by the partial derivative $\frac{\partial}{\partial z_{i\alpha}}$ for all $1 \leq i, \alpha \leq N$.

The condition given in Equation (10) can be divided into two conditions: one regarding the diagonal elements of b, the other regarding the off-diagonal elements. The diagonal condition can be written as $f(bz) = f(z)$. Examples of polynomials satisfying this condition are minors of determinants. Let $\Delta^{i_1 i_2 \ldots i_s}_{j_1 j_2 \ldots j_s}(z)$ be the determinant formed by rows $i_1 i_2 \ldots i_s$ and columns $j_1 j_2 \ldots j_s$ of the matrix z. Then $\Delta^{i_1 i_2 \ldots i_s}_{j_1 j_2 \ldots j_s}(bz) = \Delta^{i_1 i_2 \ldots i_s}_{j_1 j_2 \ldots j_s}(z)$. All elements of $V^{(m)}$ can then be written as linear combinations of products of minors of determinants. Furthermore, we shall see later that the row indices contain information about the representation, (since this action is defined to be on the left) while the column indices contain information referring to elements in the representation and thus the physical content (since this action is defined to be on the right).

The off-diagonal covariance condition can be checked one element at a time by letting the element become infinitesimally small. Then the covariant condition for element i, j can be written as:

$$(L_{i,j} f)(z) = 0; \tag{12}$$

$$L_{i,j} := \sum_{\alpha=1}^{N} z_{i\alpha} \frac{\partial}{\partial z_{j\alpha}}. \tag{13}$$

Thus the off-diagonal covariance condition can be given as those polynomials that satisfy Equation (12) for all off-diagonal entries of b. When zeros occur in (m), the setup can be simplified. Particularly, if there are p non-zero elements of (m), then only $p \times N$ complex variables are needed.

Consider next an r-fold tensor product of the vector spaces given in Equation (10), with labels $(m)_1, (m)_2 \ldots (m)_r$. Denote the product as

$$V^{(m)_1} \otimes V^{(m)_2} \otimes \ldots \otimes V^{(m)_r}. \tag{14}$$

Let p_j be the number of nonzero entries in $(m)_j$. One can then drop the zero elements and form an n-tuple of non-zero integers

$$[m] = ((m_1 \ldots m_{p_1})_1, (m_1 \ldots m_{p_2})_2 \ldots (m_1 \ldots m_{p_r})_r), \tag{15}$$

with $n = \sum_{i=1}^{r} p_i$. Let $P^{[m]}$ be the vector space of polynomials over the matrix of complex variables Z satisfying the diagonal left covariance condition:

$$P^{[m]} = (F : \mathbb{C}^{n \times N} \to \mathbb{C} \mid F(dZ) = d_{11}^{m_1} \ldots d_{nn}^{m_n} F(Z)), \tag{16}$$

where d is a diagonal $n \times n$ matrix, $d = \mathrm{diag}(d_{11} \dots d_{nn})$ and Z is an $n \times N$ matrix of complex variables, formed from the matrices of variables from each $V^{(m)_j}$ space. In physics, one often has the eight dimensional representation of $SU(3)$, denoted (2,1,0), tensored with itself. In this case $[m] = (2, 1, 2, 1)$ and Z is a 4×3 matrix of complex variables.

In this language, $SU(N)$ irreps are special cases of $U(N)$ irreps, where the last integer of the signature (m) is zero. Since all the procedures work for any values of (m), the procedures are blind to whether the irreps are from $U(N)$ or $SU(N)$, and our previous statement to just consider $U(N)$ irreps was justified.

We are interested in finding maps from the irreducible representation space $V^{(M)}$ to the product space given in Equation (14) so that Clebsch-Gordan and Racah coefficients can be computed. This can be accomplished by determining algorithmically maps from $V^{(M)}$ to $P^{[m]}$, and then consider how to restrict $P^{[m]}$ to the subspace corresponding to the product space, $V^{(m)_1} \otimes V^{(m)_2} \otimes \dots \otimes V^{(m)_r}$. The space $P^{[m]}$ can be considered as the tensor product space of representations

$$P^{[m]} = (m_1, 0 \dots 0) \otimes (m_2, 0 \dots 0) \otimes (m_n, 0 \dots 0), \tag{17}$$

considering $[m] = [m_1, m_2, \dots m_n]$. $P^{[m]}$ contains the product space as a subspace, since it meets the diagonal covariance condition, but not the off-diagonal covariance condition. The number of times the representation (M) occurs in Equation (17) is given by a Frobenius reciprocity type theorem [5]. The left diagonal condition becomes a weight condition on Gel'fand-Zetlin Tableau, and the number of tableau is then equal to the multiplicity of $V^{(M)} \subseteq P^{[m]}$.

These Gel'fand-Zetlin Tableau not only give the multiplicity in the space $P^{[m]}$ [7], but also maps between the spaces $V^{(M)}$ and $P^{[m]}$. A Gel'fand-Zetlin tableau is an array of numbers, with n-numbers on the top row, $n-1$ numbers on the next row, and subsequently one less number on each row, with the bottom row containing just one number. Denoting the ith number on the jth row as k_i^j, each tableau must satisfy the betweenness relation

$$k_i^j \geq k_i^{(j-1)} \geq k_{(i+1)}^j. \tag{18}$$

Furthermore, define the weight of each row as

$$w_j := \sum_{i=1}^{(n-j+1)} k_i^j. \tag{19}$$

The weights must be related to $[m]$ by $w_j - w_{(j-1)} = m_{(n-j+1)}$ with $w_n = k_1^n = m_1$ and $w_1 - w_2 = m_n$. To compute the multiplicity of $V^{(M)}$ in $P^{[m]}$, chose the top row related to (M) in the following way. All representations of $U(N)$ of the form $(m_1 + l, m_2 + l \dots m_N + l)$ are equivalent for any positive integer l. Thus choose the top row to be a representation equivalent to (M), such that the top row has the appropriate weight, and append zeros such that the length of the top row is n. The multiplicity of $V^{(M)}$ in $P^{[m]}$ is given by the number of tableau satisfying the above conditions.

We can consider the example of the eight-dimensional representation of $SU(3)$ crossed with itself, $[m] = (2, 1, 2, 1)$. Consider also the multiplicity of the same representation in this product, i.e. choose $(M) = (2, 1, 0)$. The weight of the top row must be six $(2 + 1 + 2 + 1 = 6)$ and include four numbers (the

same as $[m]$), and thus we choose $l = 1$ and the top row to be $(3, 2, 1, 0)$. The weight of the second row must be $6 - m_4 = 6 - 1 = 5$, etc. Thus the possible tableaux are as follows:

$$\begin{pmatrix} 3 & & 2 & & 1 & & 0 \\ & 3 & & 2 & & 0 & \\ & & 3 & & 0 & & \\ & & & 2 & & & \end{pmatrix}, \begin{pmatrix} 3 & & 2 & & 1 & & 0 \\ & 3 & & 2 & & 0 & \\ & & 2 & & 1 & & \\ & & & 2 & & & \end{pmatrix},$$
$$\begin{pmatrix} 3 & & 2 & & 1 & & 0 \\ & 3 & & 1 & & 1 & \\ & & 2 & & 1 & & \\ & & & 2 & & & \end{pmatrix}, \begin{pmatrix} 3 & & 2 & & 1 & & 0 \\ & 2 & & 2 & & 1 & \\ & & 2 & & 1 & & \\ & & & 2 & & & \end{pmatrix}. \quad (20)$$

Each tableau generate maps from $V^{(m)}$ to $P^{[m]}$, given as products of differential operators. Such operators act purely on the row indices of the minors of determinants, and thus commute with the operators acting on the column indices and do not affect the physical content. The differential operators can be read from the tableau by shifting all values to the left, placing $[m]$ as the bottom row, and then replacing any blanks by the value in the same column, bottom row. Thus the tableau in Equation (20) become

$$\begin{pmatrix} 3 & 2 & 1 & 0 \\ 3 & 2 & 0 & 1 \\ 3 & 0 & 2 & 1 \\ 2 & 1 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 2 & 1 & 0 \\ 3 & 2 & 0 & 1 \\ 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 3 & 2 & 1 & 0 \\ 3 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 2 & 1 & 0 \\ 2 & 2 & 1 & 1 \\ 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 \end{pmatrix}. \quad (21)$$

The differential operators are then directly correlated to the operations to change each row to the row below. Thus in the first tableau in Equation (21), row one is transformed to row two by moving one unit from column three to column four, corresponding to $L_{4,3}$. The subsequent transformations are two units from column three to column three, $(L_{3,2}^2)$ and one unit from column one to column two $(L_{2,1})$. Thus the first tableau in Equation (21) corresponds to the differential operator $L_{2,1}L_{3,2}^2L_{4,3}$. In general, each tableau yields a product of left differential operators we can label $\overrightarrow{(\pi \mathrm{L})}$, where $(\pi L)_i$ is the product of operators corresponding to the ith tableau. Let λ denote the number of Gel'fand-Zetlin Tableau. Then for any $f(Z) \in V^{(M)}$, $((\pi L)_i f)(Z) \in P^{[m]}$ for each i, and $(\overrightarrow{(\pi \mathrm{L})}\, f)(Z)$ is a vector of λ linearly independent vectors spanning the basis of $f(Z) \in V^{(M)} \subseteq P^{[m]}$.

In this manner both the multiplicity of $V^{(M)}$ in $P^{[m]}$ and the maps from $V^{(M)}$ to $P^{[m]}$ can be explicitly computed. However, $P^{[m]}$ only satisfies the diagonal covariant condition. Thus the product space of Equation (14) is a subset of $P^{[m]}$ and can be found by determining the appropriate linear combinations of

$(\overrightarrow{(\pi \mathrm{L})}\ f)(Z)$ such that Equation (12) is satisfied for all necessary off-diagonal operators.

To facilitate determining these linear combinations, one needs choose a polynomial $f(Z) \in V^{(M)}$ on which to act. This choice, however, must be independent of the basis for elements of $V^{(M)}$. The unique basis independent polynomial which is annihilated by all raising operators is called the highest weight vector, and such shall be our choice for f. Denote the highest weight vector $f_{\max}$. The highest weight vector of $V^{(M)}$ is given by:

$$f_{\max}(Z) = (\Delta_1^1)^{M_1 - M_2} (\Delta_{12}^{12})^{M_2 - M_3} \dots (\Delta_{12\dots N}^{12\dots N})^{M_N}, \tag{22}$$

where $\Delta_{j_1 j_2 \dots j_s}^{i_1 i_2 \dots i_s}$ is the determinant from rows $i_1 i_2 \dots i_s$ and columns $j_1 j_2 \dots j_s$ of the matrix Z . Note it is no longer necessary to explicitly notate the functional dependent on Z, and thus $\Delta_{j_1 j_2 \dots j_s}^{i_1 i_2 \dots i_s}$ is understood to be $\Delta_{j_1 j_2 \dots j_s}^{i_1 i_2 \dots i_s}(Z)$.

The off-diagonal condition, also called the Borel condition, will be solved if we determine a matrix B such that $L_{i,j}(B(\vec{\pi L}) f_{\max})(z) = 0$ for the off-diagonal elements $b_{i,j}$. The column dimension of B is λ, the multiplicity of $V^{(M)}$ in $P^{[m]}$; the row dimension is η, is the multiplicity of $V^{(M)}$ in the tensor product space. Thus solving the Borel condition and finding the matrix B not only solves the difficult problem of determining the multiplicity, but also provides a basis of the concrete representation of these multiple irreps. Once solved, one has determined $\vec{\Phi}$, where

$$\Phi_a := B_{a,k} (\pi L)_k f_{\max} \tag{23}$$

and B is such that $L_{i,j} \Phi_a = 0$ for the necessary off-diagonal i, j. Specifically, in our above example of $(M) = (3, 2, 1, 0)$ and $[m] = (2, 1, 2, 1)$, we have

$$\begin{aligned} (\pi L)_1 &= L_{2,1} L_{3,2}^2 L_{4,3}, \\ (\pi L)_2 &= L_{3,1} L_{3,2} L_{4,3}, \\ (\pi L)_3 &= L_{3,1} L_{4,2}, \\ (\pi L)_4 &= L_{3,2} L_{4,1}, \end{aligned} \tag{24}$$

and

$$\Phi_a = B_{a,k} (\pi L)_k f_{\max}. \tag{25}$$

The matrix B is defined such that $L_{1,2} \Phi_a = 0$ and $L_{3,4} \Phi_a = 0$ for all $a = 1 \dots 4$.

The computational details of solving for the Borel condition matrix will be discussed later in Section 5.1.4. Thus the above procedure determines both the multiplicity of $V^{(M)}$ in the product space $V^{(m)_1} \otimes V^{(m)_2} \otimes \dots \otimes V^{(m)_r}$, as well as determines $\vec{\Phi}$, a concrete realization of a basis of the space. Since in general the elements of $\vec{\Phi}$ are not orthogonal, the next step is to use generalized Casimir operators to determine and label orthogonal bases in the product space. Thus the orthogonal bases will can be given as linear combinations of the elements of $\vec{\Phi}$.

3.2 Racah Coefficients

Although the details of generalized Casimir operators are contained in Section 4, at this point it is helpful to note a few qualities about these operators. Specifically, generalized Casimir operators are Hermitian operators, and there exist

sets of mutually commuting operators that are both related to coupling schemes and are sufficient to break any degeneracy [4].

Consider a complete set of mutually communing generalized Casimir operators corresponding to a given coupling scheme. Enumerate the operators as $C^{(1)}, C^{(2)}, \ldots$. Since the generalized Casimir operators are Hermitian, their eigenvalues are real and the eigenvectors are orthogonal. The orthogonal basis in which all $C^{(i)}$ are diagonal is given by linear combinations of the elements of $\vec{\Phi}$. One first determines the matrix representation of the Casimir operators by letting them act on $\vec{\Phi}$. Let $\left[C^{(i)}\right]$ denote the matrix representation of the operator $C^{(i)}$ and P be the matrix that diagonalizes all $\left[C^{(i)}\right]$. By construction $P^{-1}\left[C^{(i)}\right]P$ is diagonal for any i, and thus

$$C^{(i)}P^{-1}\vec{\Phi} = P^{-1}C^{(i)}\vec{\Phi} = P^{-1}\left[C^{(i)}\right]\vec{\Phi} = \left(P^{-1}\left[C^{(i)}\right]P\right)P^{-1}\vec{\Phi}, \tag{26}$$

where $C^{(i)}$ commutes with P^{-1} since $C^{(i)}$ is a differential operator and P is a matrix of constant coefficients. $P^{-1}\vec{\Phi}$ is then the orthogonal basis corresponding to the given coupling scheme.

Denote two coupling schemes by A and B. There exists two copmlete sets of generalized Casimir operators, $\{C_A^{(i)}\}$ and $\{C_B^{(i)}\}$, that both specify the respective coupling schemes and label the basis vectors. Let $P_{(A)}$ and $P_{(B)}$ be the matrices that diagonalize all the operators for each respective coupling scheme. Each element of the basis $P_{(A)}^{-1}\vec{\Phi}$ can be labeled by the eigenvalues of the set $\{C_A^{(i)}\}$, and likewise with coupling scheme B. Let (η^A) denote one of the sets of eigenvalues labeling a basis element in coupling scheme A, and likewise let (η^B) denote one of the sets of eigenvalues for coupling scheme B. Thus the two sets of basis vectors for $V^{(M)}$ in the product space can be written as $\left|(M),(\eta^A)\right\rangle$ and $\left|(M),(\eta^B)\right\rangle$. Racah coefficients are just the overlap between the two orthogonal basis, and can now be written succinctly as the matrix:

$$R = \left\langle (M),(\eta^A)\right|\left.(M),(\eta^B)\right\rangle. \tag{27}$$

Note that the above equation is the generalization of Equation (8), accounts for the multiplicity, and allows for any general coupling scheme of any r-fold product.

3.3 Gel'fand-Zetlin Basis

One of the main challenges of computing Clebsch-Gordan coefficients for general r-fold tensor products is the lack of a unique basis for $U(N)$ with $N > 2$. The Gel'fand-Zetlin basis has long been described as a general basis for $U(N)$, but in the past it was not known how to directly compute the basis elements. Thus the following algorithms to compute the basis are themselves a triumph, besides providing a basis in which to compute Clebsch-Gordan coefficients.

One should not confuse this basis with the basis related to Racah coefficients. Suppose a given irrep (M) occurs η times in the direct sum basis. The Casimir Operators related to the signature (M) are all η-fold degenerate over the space the multiple (M) irreps, and thus do not separate the multiple occurrences. Generalized Casimir operators related to coupling schemes yields a basis among this degeneracy and thus can separate and label the η different occurrences of

(M). This is the basis discussed above in connection with Racah coefficients, and can be thought of as an "external" basis. However, within each irrep are various elements, and thus there exists a choice of basis among these internal elements. This internal basis is the one we choose to be the Gel'fand-Zetlin basis. To specify an element in the direct sum basis, one must both specify which occurrence of the irrep (M) (labeled by generalized Casimir operators) as well as which element in the irrep (labeled by a Gel'fand-Zetlin Tableau). Part of the beauty of the procedures outlined in this paper is that all "external" considerations regard the row indices of the minors, while all "internal" considerations regard the column indices. Thus both can be considered separately as they act on completely different spaces, and yet the action of the one on the row indices is identical to the action of the other on the column indices.

The Gel'fand-Zetlin basis is precisely that basis in which each basis element of an irrep (M) is labeled by a Gel'fand-Zetlin tableau. This correspondence is one-to-one provided the following conditions. For the $U(N)$ irrep (M), each tableau contains N rows, the top row being the N-tuple (M) and the bottom row being a single integer. Each row must obey the dominant condition and the betweenness relations given in Equations (9) and (18). There is no weight condition at present, since we are considering all elements and thus all possible weights. The meaning of each row is related to considering $U(N) \supseteq U(N-1) \supseteq \ldots \supseteq U(1)$. The row with N' entries is the irrep signature of the $U(N')$ subgroup of the $U(N)$ irrep.

For $SU(2)$, we have signature $(M) = (2j, 0)$. Thus there would be tableau

$$\begin{pmatrix} 2\,j & & 0 \\ & k & \end{pmatrix}, \tag{28}$$

and the conditions will be satisfied for $0 \leq k \leq 2j$. This can be related to the usual $|j, m\rangle$ basis by letting $k = 2(m + j)$. Note there are $2j + 1$ tableau, as expected, corresponding to the $2j + 1$ dimensions of the irrep j.

For $(2, 1, 0)$, the eight-dimensional irrep of $SU(3)$, there are eight tableau that meet the conditions. For example, three of these tableaux are:

$$\begin{pmatrix} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 2 & & \end{pmatrix}, \begin{pmatrix} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 1 & & \end{pmatrix}, \begin{pmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 0 & & \end{pmatrix}. \tag{29}$$

To compute the representation of the Gel'fand-Zetlin in our representation space $V^{(M)}$, we begin by considering the highest weight vector. Since this vector in unique, we can associate the highest weight vector in our representation with the highest weight tableau. The highest weight vector in $V^{(M)}$ is given by Equation (22). To recognize the highest weight vector, recall the earlier discussion relating the changes between rows with lowering operators. The highest weight tableau corresponds to the tableau with no lowering–each row is just the row above with the last element removed. For $(2, 1, 0)$ the highest weight tableau is the first tableau listed in Equation(29) and the highest weight vector in $V^{(2,1,0)}$ is $\Delta^1_1 \Delta^{1,2}_{1,2}$.

Let h denote a certain tableau in a $U(N)$ irrep (M). We want to determine a representation of h in $V^{(M)}$, which shall denoted as $h(Z)$. Let $(w) = (w_1, w_2 \ldots w_N)$ denote the weights of h, defined in Equation (19). The

desired basis element is contained in the space of (M) embedded in the tensor product $(w_1, 0, \ldots 0) \otimes (w_2, 0, \ldots 0) \otimes (w_N, 0, \ldots 0)$. Again we construct all tableau with the desired weight structure and read off the lowering operators corresponding to each tableau. The only difference in this procedure is that these operators are right operators acting on the column indices rather than left operators acting on row indices, but this is just a matter of bookkeeping. We then apply each product of lowering operators to the highest weight vector in our $V^{(M)}$ space. Since each $(w_i, 0, \ldots 0)$ only has one non-zero element, there are no off-diagonal elements and thus no Borel condition to solve. We now have $\vec{\Phi}' = \overrightarrow{(\pi \mathrm{R})}\ f_{\max}(Z)$ as a set of linearly independent vectors spanning the (column) space of (M) in $(w_1, 0, \ldots 0) \otimes (w_2, 0, \ldots 0) \otimes (w_N, 0, \ldots 0)$.

The next step is to pick an orthogonal basis. In this case, we will just be using traditional Casimir operators, since we are working internal in the space $V^{(M)}$. The set of mutually commuting Casimir operators is fixed by the by the formula relating the eigenvalues of Casimir operators to the tableau elements [10]. Thus one diagonalizes with respect to the set which corresponds to the tableau elements, which in turn determines the orthogonal basis. Each element of this basis will correspond to a tableau with the same weight as the one of interest. Using the formula to relate the eigenvalues with the tableau elements, one can readily observe which polynomial representation corresponds with which tableau. In this manner, all Gel'fand-Zetlin basis elements can be computed for any $U(N)$ irrep space.

3.4 Clebsch-Gordan Coefficients

We are finally in a position to discuss Clebsch-Gordan coefficients for general r-fold $U(N)$ tensor products. First, one should review notation. Previously, we considered Clebsch-Gordan coefficients of the form in Equation (4). This generalizes to,

$$\left\langle (M), h, (\eta^A) \,|(m_1), h_1\rangle \,|(m_2), h_2\rangle \ldots |(m_r), h_r\rangle \right., \tag{30}$$

where (M), $(m)_i$ are the irrep labels; h, h_i are the tableau specifying a basis element in each space; and (η^A) are the eigenvalues labeling the multiplicity, corresponding to a coupling scheme A. Since we have concrete polynomial representations for both the left and right side of this inner product, all that remains is just to choose the best computational method for carrying out the inner product. The author's implementation of the procedures returns a vector, the ith element corresponding to the ith set of values (η^A).

One should note that in order for the coefficient to be non-zero, then for every jth row, the sum of the jth row in tableau h must correspond to the sum of all the jth rows of all the h_i. For $SU(2)$ this reduces to the well known rule of thumb that the total m value is the sum of the m values being combined.

4 Generalized Casimir Operators

4.1 Background and Notation

A brief aside into the details of generalized Casimir operators is of worth. More explicit consideration is given in [1] and [6]. Traditional Casimir operators are

defined as Hermitian operators which are polynomials in a given Lie algebra and commute with all elements of the algebra. Thus Casimir operators are always diagonal in any representation. Furthermore, Casimir operators are multiplies of the identity if and only if the representation is irreducible. The number of linearly independent Casimir operators is known as the rank of the Lie algebra. For $U(N)$, the rank is N and the values in the N-tuple (M) are directly related to the eigenvalues of the N Casimir operators.

In the representation spaces outlined in Section 3.1, Casimir operators can be written as products of the left differential operators $L_{i,j}$, as in Equation (13). Note, that since we are considering $V^{(M)}$ contained in the product space $V^{(m)_1} \otimes V^{(m)_2} \otimes \ldots \otimes V^{(m)_r}$, all traditional Casimir operators from any of the spaces will be multiples of the identity. generalized Casimir operators are similarly defined as Hermitian operators which are polynomials in the Lie algebra elements and commute with all the Lie algebra elements. However, generalized Casimir operators mix multiple irreps in the product space, rather than acting solely on single irrep spaces $V^{(m)_i}$. It has been proven that the there exist sufficient generalized Casimir operators to fully break any degeneracy in the product space [4].

Consider again the general product space $V^{(m)_1} \otimes V^{(m)_2} \otimes \ldots \otimes V^{(m)_r}$. Polynomials in this space will be over the variables specified by a matrix Z. Note that Z has $n = p_1 + p_2 + \ldots + p_r$ rows, as in Equation (15). Also note that the first p_1 rows corresponds to $V^{(m)_1}$, the next p_2 rows correspond to $V^{(m)_2}$, and so forth. All possible left differential operators $L_{i,j}$ given in Equation (13) can be considered as part of an array, with each of i and j ranging from 1 to n. This array can be divided into $r \times r$-blocks. Denote blocks by $[L]_{i,j}$, where each block $[L]_{i,j}$ is p_i rows by p_j columns. Thus $[L]_{i,i}$ contain all of the left differential operators acting purely on space $V^{(m)_i}$, whereas block $[L]_{i,j}$ mixes from space $V^{(m)_i}$ to space $V^{(m)_j}$. It has been shown [7] that operators given as the trace of sums of products of such blocks are generalized Casimir operators, if such operators are Hermitian and if each product has the form

$$[L]_{i_1,i_2}[L]_{i_2,i_3} \ldots [L]_{i_d,i_1} \tag{31}$$

for a product of d blocks. It can be shown that the adjoint of such an operator is obtained by inverting the order of the indices in each product. Thus one can easily test whether an operator in this form is Hermitian.

For example, consider the three fold tensor product with $(m)_1 = (m)_2 = (m)_3 = (2, 1, 0)$, and $(M) = (4, 3, 2)$. As previously discussed, this is the eight-dimensional representation of $SU(3)$ sitting in the 3-fold tensor product of it with itself. Then $p_1 = p_2 = p_3 = 2$ and $n = 6$. Thus the array of all $L_{i,j}$ is a $p \times p = 6 \times 6$ array, divided into a $r \times r = 3 \times 3$ array of 2×2 blocks. The following are all generalized Casimir operators:

$$\begin{aligned} &[L]_{1,2}[L]_{2,1}, \\ &[L]_{1,2}[L]_{2,2}[L]_{2,1}, \\ &[L]_{1,2}[L]_{2,3}[L]_{3,1} + [L]_{1,3}[L]_{3,2}[L]_{2,1}. \end{aligned} \tag{32}$$

Note that the operator $[L]_{1,2}[L]_{2,3}[L]_{3,1}$ is not Hermitian, but its adjoint is $[L]_{1,3}[L]_{3,2}[L]_{2,1}$ and thus the sum $[L]_{1,2}[L]_{2,3}[L]_{3,1} + [L]_{1,3}[L]_{3,2}[L]_{2,1}$ is Hermitian.

A generalized Casimir operator is said to be of degree d if each product of blocks contains exactly d blocks. This is equivalent to stating that the operator is the sum of products of exactly d left differential operators. For traditional Casimir operators in $U(N)$, operators with degree greater than N can be expressed as linear combinations of Casimir operators with degrees less than or equal to N. For generalized Casimir operators, it is also known that there exists an upper bound $\tilde{N}$, of which operators with degree greater than $\tilde{N}$ are not linearly independent from those of degree equal or less than $\tilde{N}$. In practice, often $\tilde{N} = N$, but the exact relation in general has yet to be proven. Since there exists an upper bound, methods based on iteratively increasing the degree will eventually break all degeneracy as long as a sufficient number of operators of each degree are considered.

4.2 Commutation Relations

Based on the definition given in Equation (13), one can easily verify the following result:

$$[L_{i,j}, L_{k,l}] = \delta_{j,k} L_{i,l} - \delta_{i,l} L_{k,j}. \tag{33}$$

Since the set of row indices (which the $L_{i,j}$ act on) are disjoint for each space $(m)_k$, the above equation equivalently holds for blocks $[L]_{i,j}$. Note that each product in an arbitrary generalized Casimir operator of degree d can also be written as:

$$\begin{aligned} &\mathrm{tr}\left([L]_{u_1,u_2}[L]_{u_2,u_3}\dots[L]_{u_d,u_1}\right) \\ &\quad = \sum_{i_1\in u_1}\dots\sum_{i_2\in u_2}\dots\sum_{i_d\in u_d} L_{i_1,i_2}L_{i_2,i_3}\dots L_{i_d,i_1}, \end{aligned} \tag{34}$$

where in the first sum i_1 ranges over the p_{u_1} values corresponding to $(m)_{u_1}$, and so forth.

A few special commutation relations are worthy of note:

$$[L_{k,k}, L_{i,j}] = L_{i,j}(\delta_{i,k} - \delta_{i,j}) \tag{35}$$

$$[(L_{k,k})^a, L_{i,j}] = L_{i,j}\left((L_{k,k} + \delta_{i,k} - \delta_{i,j})^a - (L_{k,k})^a\right) \tag{36}$$

$$[(L_{k,k})^a, C] = 0. \tag{37}$$

The C in Equation (37) is any generalized Casimir operator of the form described in the text above Equation (31).

4.3 The Binary Coupling Method

This section presents two versions of the author's Binary Coupling Method. Although these methods do not work in general, in practice they are most useful. The method is based on representing each parenthesis in a coupling scheme by operators which are sums of generalized Casimir operators over two spaces. Any generalized Casimir operator over two spaces, say i_1, i_2, is equivalent to an operator of the form:

$$C^{(a,b,c)}_{i_1,i_2} = \mathrm{tr}\left([L]_{i_1,i_2}\left([L]_{i_2,i_2}\right)^a[L]_{i_2,i_1}\left([L]_{i_1,i_2}[L]_{i_2,i_1}\right)^b\left([L]_{i_1,i_1}\right)^c\right) + \text{adj.} \tag{38}$$

where "+adj." refers to adding the adjoint of the first term. For each degree d and each parenthesis in the coupling scheme, the General Binary Coupling

method adds to the mutually commuting set all operators of the above form such that $a + 2b + c = d - 2$. The Special Binary Coupling method uses only one operators per parenthesis: the one corresponding to the choice $b = c = 0$ and thus $a = d - 2$. Specifically, these operators are of the form:

$$C_{i_1,i_2} \quad = \quad \mathrm{tr}\left([L]_{i_1,i_2} \left([L]_{i_2,i_2} \right)^{d-2} [L]_{i_2,i_1} \right). \tag{39}$$

One can easily verify that all operators of the form in Equation (38) commute for various values of a, b, c. Much more difficult is the proof that operators of a certain choice of a, b, c are no more degenerate than certain other choices of a, b, c. In practice, however, operators of the form in Equation (39) are sufficient. Thus only the special case is presented below, as the generalization is obvious once the specific case is presented.

A given coupling scheme of r-objects can be denoted by $r - 1$ parentheses. Each parenthesis is a binary operation coupling two sets of spaces. Enumerate the parentheses, and define the sets s_i, s_i' such that the ith parenthesis couples the spaces s_i on the right to the spaces s_i' on the left. For each parenthesis, a generalized Casimir operator of a given degree $d = a + 2$ is chosen as follows:

$$C_{(i)} := \mathrm{tr}\left(\sum_{j \in s_i} \sum_{k \in s_i'} [L]_{j,k} \left([L]_{k,k} \right)^a [L]_{k,j} \right). \tag{40}$$

One can easily verify that all such operators form a mutually commuting set. The general method would be to begin with one operator per parenthesis of degree 2, and then iteratively increase the commuting set to include operators for each parenthesis of a consecutively higher degree. However, in practice, it is generally sufficient to choose $a+2 = d = N$, where the irrep (M) is of the group $U(N)$. The author's computer implementation of these methods allow the user to input which degree(s) to use, defaulting to $d = N$ if no degree is specified.

To prove that either of these methods is sufficient to generate a complete set of operators, several additional steps would be needed. One would need prove that it is sufficient to use solely binary couplings, and that particular choices of a, b, c in the General Binary Method are no more degenerate than the other choices. Also assumed in the method is that it is sufficient to use generalized Casimir operators that are all sums of products of the same degree. While such proofs are yet to be shown, no counter examples have been found to date.

4.4 The Complete Coupling Method

The method of this section is only useful for $SU(2)$, and is generally not sufficient for higher $U(N)$ irreps. Since all blocks of operators $[L]_{i,j}$ are 1×1, all $[L]_{i,j} = L_{i,j}$. Thus all generalized Casimir operators acting on the product space are analogous to traditional Casimir operators acting on a (not necessarily reducible) $U(r)$ irrep. Thus the eigenvalues are integers and can be related to the usual m intermediate values by known formulas.

Consider an enumeration of the parenthesis, and define s_i to be the set of all spaces enclosed by the ith parenthesis. The operator of degree d representing the ith parenthesis is then chosen to be

$$C_{(i)} := \mathrm{tr}\left(\sum_{k_1 \in s_i} \sum_{k_2 \in s_i} \cdots \sum_{k_d \in s_i} [L]_{k_1,k_2} [L]_{k_2,k_3} \ldots [L]_{k_d,k_1} \right). \tag{41}$$

This method of realizing coupling schemes is also programmed in the author's computer implementation.

5 Computational Considerations

5.1 Mapping the Highest Weight Vector

5.1.1 Differential Operators

To consider the computer implementation of these procedures, a few details regarding the action of the differential operators on the minors of determinants is of worth. We shall choose a canonical ordering of the rows and columns such that they are always listed in increasing order on each minor of determinant. Let the size of a minor determinant be defined as the number of rows (equivalently columns) it involves, and define the structure of a product of minors of determinants as the number of minors of each size in the product. Thus the minor $\Delta^{1,2,3}_{4,5,6}$ would have size three and the product $\Delta^1_1\Delta^{2,3}_{1,2}\Delta^{1,3,4}_{1,2,3}\Delta^{2,5,6}_{1,2,3}$ has the structure of one minor each of size one and two and two minors of size three.

Consider then some $L_{i,j}$ acting on a single minor of determinant. If the determinant does not include row j, then the result is zero, as the expression is then constant with respect to all the partial derivatives. Suppose then that j is one of the rows listed. Note that $L_{i,j}$ has the effect of changing $z_{j,\alpha}$ to $z_{i,\alpha}$, for each value of α, thus changing an index of the minor of determinant from j to i. Permuting the list of indices to a canonical ordering may introduce a phase of -1, as minors of determinants are antisymmetric with interchange of any two indices. Note, if i was already included in the list of indices, the result is also zero, as the result is a minor over duplicate rows. Thus an operator $L_{i,j}$ either annihilates a minor of determinant or introduces a phase of ± 1 and replaces j with i in its list of indices. Operators $L_{i,j}$ act on products of minors of determinants by the typical chain rule, resulting in a sum of products: each term generated from the operator acting solely on one of the minors in the product. Right operators $R_{i,j}$ act identically on the column indices.

5.1.2 Spanning $V^{(M)}$

It readily observed that these differential operators acting on a product of minors of determinants will yield a linear combination of products with the same structure as the initial product. $V^{(M)}$ is a realization of the $U(N)$ irrep (M), and thus all elements of $V^{(M)}$ can be obtained by applying lowering operators to the highest weight vector. In this manner we see that all elements of $V^{(M)}$ can be written as linear combinations of products of minors with the same structure. Thus the enumeration of all such possible products with the same structure as the highest weight vector span the space $V^{(M)}$. In most of the procedures, we are solely acting on either the row or column indices. Thus in each case we will only enumerate all the possibilities for one or the other, rather than computing all the possible combinations of both. The recombining of the indices will be left to the only place necessary, the final portion of the Clebsch-Gordan coefficient procedure.

We can yet find a smaller set that still spans the space by observing considering the weight of $[m]$. Note that the highest weight vector in $V^{(M)}$, Equation

(22), has each row index i included M_i times, similar to the ith column of the top row of the tableau being M_i. Each tableau corresponds to a different way of lowering from (M) to $[m]$. Note though that for each unit from column j to i changed on a tableau yields an operator $L_{i,j}$, which has the effect of changing a j to an i on the product of minors of determinants. Thus, one can easily see that all products of minors of determinants in $\overrightarrow{(\pi \mathrm{L})}\, f_{\max}$ not only has the same structure as $f_{\max}$, but has each row i included m_i times. Thus one can easily construct a spanning set of products of minors of determinants, and each $(\pi L)_i f_{\max}$ can be written as a vector of constant coefficients multiplied by the spanning set. Denote an enumeration of this spanning set as $\vec{\Delta}$. Note that this set $\vec{\Delta}$ can be constructed explicitly just from knowing (M) and $[m]$, without actually applying any operators.

For example, embedding (2,1,0) in $(2,1,0)\otimes(2,1,0)$ we have $(M) = (3,2,1,0)$ and $(m) = (2,1,2,1)$. Since $M_1 - M_2 = M_2 - M_3 = M_3 - M_4 = 1$, all products will be a minor over one index, a minor over two indices, and a minor over three indices. Also, each product will include two 1's, one 2, two 3's, and one 4. Computing the possibilities yields the following eight products of minors of determinants:

$$\vec{\Delta} = \begin{bmatrix} \Delta^{1}\Delta^{1,3}\Delta^{2,3,4} \\ \Delta^{1}\Delta^{2,3}\Delta^{1,3,4} \\ \Delta^{1}\Delta^{3,4}\Delta^{1,2,3} \\ \Delta^{2}\Delta^{1,3}\Delta^{1,3,4} \\ \Delta^{3}\Delta^{1,2}\Delta^{1,3,4} \\ \Delta^{3}\Delta^{1,3}\Delta^{1,2,4} \\ \Delta^{3}\Delta^{1,4}\Delta^{1,2,3} \\ \Delta^{4}\Delta^{1,3}\Delta^{1,2,3} \end{bmatrix}. \tag{42}$$

Although this is yet a small set for $(2,1,0) \subseteq (2,1,0) \otimes (2,1,0)$, one can consider the 3, 4 and 5-fold product of $(2,1,0)$ with itself, and consider the multiplicity of the irrep (2,1,0) in this product. This spanning set $\vec{\Delta}$ quickly gets very large. For the 2-fold case, it has dimension 8; 3-fold, 168; 4-fold, 6,100; and in the 5-fold case, over 250,000. Thus efficiency in generating, enumerating, and storing $\vec{\Delta}$, is one of the critical issues in optimizing these procedures for larger products.

5.1.3 Mapping to $P^{[m]}$

We can now consider the concrete mapping of the highest weight vector of $V^{(M)}$ into the tensor product space $V^{(m)_1} \otimes V^{(m)_2} \otimes \ldots \otimes V^{(m)_r}$. The computer easily constructs the Gel'fand-Zetlin Tableau and reads off the lowering operators $\overrightarrow{(\pi \mathrm{L})}$ The highest weight vector in $V^{(M)}$ is then constructed by Equation (22). The next step is to apply $\overrightarrow{(\pi \mathrm{L})}$ to $f_{\max}$. The results are stored as the matrix $\tilde{M}$ defined such that $(\pi L)_i f_{\max} = \tilde{M}_{i,j}\Delta_j$, or more succinctly as

$$\overrightarrow{(\pi \mathrm{L})}\, f_{\max} = \tilde{M}\Delta. \tag{43}$$

5.1.4 Borel Condition

The next step in mapping the highest weight vector is to satisfy the Borel, or off-diagonal covariance, condition. Enumerate the off-diagonal operators as $L^{(1)}, L^{(2)}, \ldots L^{(last)}$. The goal is then to determine a matrix B such that

$$L^{(i)} B \overrightarrow{(\pi \mathrm{L})}\, f_{\max} = \vec{0}, \tag{44}$$

for all the necessary $L^{(i)}$. The most efficient method is to compute the matrix B iteratively.

First, consider some unknown linear combination given by vector $\vec{c}$ such that

$$\begin{aligned} \vec{0} &= L^{(1)} \left(c^T \overrightarrow{(\pi \mathrm{L})}\, f_{\max} \right) \\ &= L^{(1)} \left(c^T \tilde{M} \vec{\Delta} \right) \\ &= c^T \tilde{M} \left(L^{(1)} \vec{\Delta} \right) \\ &= c^T \tilde{M} A \vec{\Delta}' \\ &\Rightarrow (\tilde{M} A)^T \vec{c} = \vec{0}, \end{aligned} \tag{45}$$

where A and $\vec{\Delta}'$ are defined such that $L^{(1)} \Delta_i = A_{i,j} \Delta'_j$. Also note that differential operators such as $L^{(1)}$ commute with the matrices and vectors of constant coefficients, and only act on the basis of products of minors of determinants. This final expression for $\vec{c}$ in the bottom line of the above equation is underdetermined, and thus $\vec{c}$ can be written as a matrix multiplied by a vector of free variables. Let the transpose of this matrix be denoted as $B^{(1)}$. Since Equation (45) is true regardless of the values of the variables, it follows that $B^{(1)} \tilde{M} A = 0$. One could equivalently state that the transpose of the matrix $B^{(1)}$ spans the nullspace of $(\tilde{M} A)^T$. The efficient computation of the nullspace is another important issue in optimizing the program for larger products. We then have:

$$\begin{aligned} L^{(1)} \left(B^{(1)} \overrightarrow{(\pi \mathrm{L})}\, f_{\max} \right) &= L^{(1)} B^{(1)} \tilde{M} \vec{\Delta} \\ &= B^{(1)} \tilde{M} \left(L^{(1)} \vec{\Delta} \right) \\ &= B^{(1)} \tilde{M} A \vec{\Delta}' \\ &= \vec{0}. \end{aligned} \tag{46}$$

Similarly, one can consider $\vec{c}^{(2)}$ such that $L^{(2)} \left(c^{(2)T} B^{(1)} \overrightarrow{(\pi \mathrm{L})}\, f_{\max} \right) = 0$. As above, this equation is underdetermined and yields a matrix of constant coefficients $B^{(2)}$ satisfying $L^{(2)} \left(B^{(2)} B^{(1)} \overrightarrow{(\pi \mathrm{L})}\, f_{\max} \right) = 0$. In general, one can determine the matrix $B^{(i)}$ from the ith iteration by solving

$$L^{(i)} \left(\vec{c}^{(i)T} B^{(i-1)} \ldots B^{(1)} \overrightarrow{(\pi \mathrm{L})}\, f_{\max} \right) = 0, \tag{47}$$

for $\vec{c}^{(i)}$.

Note that at each iteration the matrix of coefficients of $\vec{c}^{(i+1)}$, namely $(B^{(i)}B^{(i-1)}\dots B^{(1)}\tilde{M}A^{(i)})$ has fewer rows than the previous matrix $(B^{(i-1)}B^{(i-2)}\dots B^{(1)}\tilde{M}A^{(i)})$, and thus computing the nullspace is simpler at each iteration.

Let $B := B^{(last)}\dots B^{(2)}B^{(1)}$ be the product of all the matrices. Note then that for any $L^{(i)}$

$$\begin{aligned} L^{(i)}\left(B\tilde{M}\vec{\Delta}\right) &= L^{(i)}B^{(last)}\dots B^{(2)}B^{(1)}\tilde{(M)}\vec{\Delta} \\ &= B^{(last)}\dots B^{(i+1)}L^{(i)}B^{(i)}\dots B^{(2)}B^{(1)}\tilde{(M)}\vec{\Delta} \\ &= B^{(last)}\dots B^{(i+1)}\vec{0} \\ &= \vec{0}. \end{aligned} \tag{48}$$

Define $M := B\tilde{M}$, and thus $\vec{\Phi} = M\vec{\Delta} = B\overrightarrow{(\pi \mathrm{L})}f_{\max}$ is a concrete basis for $f_{\max}$ mapped into the product space. Note then that the dimension of $\vec{\Phi}$ is the multiplicity of $V^{(M)}$ in the product space $V^{(m)_1}\otimes V^{(m)_2}\otimes\dots\otimes V^{(m)_r}$, which is computed explicitly by the above procedures.

5.1.5 Applying Generalized Casimir Operators

Let C be one of the operators in a complete set of mutually commuting generalized Casimir operators related to some coupling scheme. We then have

$$\begin{aligned} C\vec{\Phi} &= CM\vec{\Delta} \\ &= MC\vec{\Delta} \\ &= M\Gamma\vec{\Delta} \\ &= (M\Gamma M^{-1})(M\vec{\Delta}) \\ &= [C]\vec{\Phi}, \end{aligned} \tag{49}$$

where Γ is defined such that $C\Delta_i = \Gamma_{i,j}\Delta_j$, and where $[C]$ is defined as $[C] = M\Gamma M^{-1}$. Since the generalized Casimir operators are all Hermitian, $[C]$ can be diagonalized. That is, there exists P and η, (η being the diagonal matrix of eigenvalues) such that $[C] = P\eta P^{-1}$. Then Equation (49) can be written as $C\vec{\Phi} = P\eta P^{-1}\vec{\Phi}$ or

$$C\left(P^{-1}\vec{\Phi}\right) = \eta\left(P^{-1}\vec{\Phi}\right), \tag{50}$$

and thus C is diagonal is this basis.

If η is degenerate, then one can apply additional generalized Casimir operators. Similar to the above method, one can then compute a matrix P such all of the operators in the mutually commuting set are diagonal with respect to the basis $P^{-1}\vec{\Phi} = P^{-1}M\vec{\Delta}$. Each element of $P^{-1}\vec{\Phi}$ can then be labeled by eigenvalues of the chosen set of operators. Since these operators are Hermitian, the eigenvalues are real and the eigenvectors $P^{-1}\vec{\Phi}$ are orthogonal. The elements in $P^{-1}\vec{\Phi}$ can then be notated as $|(M), f_{\max}, (\eta)\rangle$, where (η) are the eigenvalues of the set of generalized Casimir operators labeling the particular basis element.

Thus, the basis which diagonalizes the set of generalized Casimir operators related to a coupling scheme is the orthogonal basis representing that coupling scheme, and the sets of eigenvalues of the operators provide labels for the basis elements. Specific examples are included in the Section 6.

5.2 Racah Coefficients

Consider then two complete sets of mutually commuting generalized Casimir operators corresponding to two coupling schemes. Note, in general, that the sets do not commute with each other. Denote the schemes as primed and unprimed. The Racah coefficients of Equation (27) can then be written alternately in Dirac or inner product notation as

$$R^{(M)}_{(\eta),(\eta')} = \langle (M) f_{\max}(\eta) \, | (M) f_{\max}(\eta') \rangle \tag{51}$$

$$= \frac{\left(P^{-1}_{(\eta)} \vec{\Phi}, P'^{-1}_{(\eta')} \vec{\Phi} \right)}{\| P^{-1}_{(\eta)} \vec{\Phi} \| \| P'^{-1}_{(\eta')} \vec{\Phi} \|}, \tag{52}$$

where P and P' are the matrices used for diagonalizing the generalized Casimir operators in the prime and unprimed coupling schemes, and the subscripts (η) and (η') indicate using just the column corresponding to the eigenvector labeled by that eigenvalue.

The numerator and denominator are computed separately. All inner products are computed by taking the adjoint of $(\overrightarrow{\pi \mathrm{L}})$. For example, consider the numerator in index notation. The overlap between the ith and jth elements is

$$\begin{aligned} \left((P^{-1}\vec{\Phi})_i, (P'^{-1}\vec{\Phi}) \right) &= \left((P^{-1}B)_{i,k} (\pi L)_k f_{\max}, (P'^{-1}M)_{j,l} \Delta_l \right) \\ &= (P^{-1}B)_{i,k} (P'^{-1}M)_{j,l} \left(f_{\max}, (\pi L)^\dagger_k \Delta_l \right) \\ &= (P^{-1}B)_{i,k} (P'^{-1}M)_{j,l} Q_{k,l} \| f_{\max} \|^2 \\ &= (P^{-1} B Q^T M^T P'^{-T})_{i,j} \| f_{\max} \|^2, \end{aligned} \tag{53}$$

where $-T$ denotes inverse transpose, and Q is defined such that $Q_{k,l} f_{\max} = (\pi L)^\dagger_k \Delta_l$. Equation (51) can now be written as

$$R^{(M)}_{(\eta),(\eta')} = \frac{P^{-1}_{(\eta)} B Q M^T P'^{-T}_{(\eta')}}{\sqrt{P^{-1}_{(\eta)} B Q M^T P^{-T}_{(\eta)}} \sqrt{P'^{-1}_{(\eta')} B Q M^T P'^{-T}_{(\eta')}}}. \tag{54}$$

Notice that the norms of $f_{\max}$ have all cancelled, and thus the method is truly basis independent, as it should be. The author's computer procedures computes the matrix Q and then uses Equation (54) to compute the coefficients. Note this is an equation for matrix elements, while the computer procedures return the entire matrix. Also note in the equation, the matrices B, M, and Q depend only on the irrep and the product space, while the matrices P and P' depend only on the coupling scheme.

5.3 Gel'fand-Zetlin Basis

Computing the Gel'fand-Zetlin basis utilizes the same routines as mapping the highest weight vector into the product space. Given a tableau corresponding to an element in $V^{(M)}$, the procedure constructs all tableau with the same weight, yielding some $(\overrightarrow{\pi \mathrm{R}})$, in analogy with Section (3.1). There is no Borel condition, and so the basis for the multiplicity is given by $(\overrightarrow{\pi \mathrm{R}}) \; f_{\max}$. Again, we can

enumerate the possible combinations of column indices, and write $\vec{\Phi}^{(c)} := \overrightarrow{(\pi \mathrm{R})}$ $f_{\max} = M^{(c)}\vec{\Delta}^{(c)}$. This is in analogy to Equation (23), except we have decorated each symbol with superscript (c) to distinguish between these and the analogous objects in Section (3.1).

The next step is to apply Casimir operators to determine the correct combination of the $\vec{\Phi}^{(c)}$ that corresponds to the given tableau. The procedure intelligently chooses which Casimir Operators are necessary to break the degeneracy, based on the relationship between the eigenvalues and the entries of the tableaux. The operators are applied iteratively, and at each iteration, the procedure restricts itself to the subspace with the appropriate eigenvalue. The result then is some particular combination, say $\vec{h}'$, of the basis elements

$$\vec{h}' M'^{(c)} \vec{\Delta}^{(c)} = \vec{h}^{(M)} \vec{\Delta}^{(c)}, \tag{55}$$

where $\vec{h}^{(M)}$ is defined as $\overrightarrow{\mathrm{h}}^{(M)} = \vec{h}' M'^{(c)}$.

The basis elements for the $V^{(m)_i}$ spaces are computed equivalently, with care to ensure that each covers the appropriate range of variables. The results can then be written as $\vec{h}^{(M)}\vec{\Delta}^{(c)}$ for the element of $V^{(M)}$ and $\vec{h}^{(i)}\vec{\Delta}^{(c,i)}$ for each $V^{(m)_i}$.

5.4 Clebsch-Gordan Coefficients

Let h denote a specific tableau, corresponding to an element in the irrep M. The concrete realization of $\left|(M), h, (\eta^{(A)})\right\rangle$ is then given by the combination of $P^{-1}M\vec{\Delta}^{(r)}$ and $\vec{h}^{(M)}\vec{\Delta}^{(c)}$, where the former is from mapping $V^{(M)}$ into the product space, and the latter from lowering the highest weight element to the element corresponding to h. Here the two $\vec{\Delta}$ are decorated with (c) and (r) to denote whether it corresponds to possible row or the column indices. The combination can then be written as

$$\left|(M), h, (\eta^{(A)})\right\rangle \rightarrow h(Z) = P^{-1} M \Delta^{(r,c)} \vec{h}^{(M)}, \tag{56}$$

where $\Delta^{(r,c)}$ is a matrix defined by $\Delta^{(r,c)}_{i,j}$ being a weighted sum of the various combinations of $\Delta^{(r)}_i$ and $\Delta^{(c)}_j$. The weights are the ratios between the number of ways to combine $\Delta^{(r)}_i$ and $\Delta^{(c)}_j$ to get that particular minor of determinants and the total number of combinations. Some examples are in order. For simplicity, consider just minors over two indices. The following combine as

$$\begin{aligned}
\Delta^{1,2} \,\&\, \Delta_{1,2} &\rightarrow \Delta^{1,2}_{1,2} \\
\Delta^{1,2}\Delta^{3,4} \,\&\, \Delta_{1,2}\Delta_{3,4} &\rightarrow \frac{1}{2}\Delta^{1,2}_{1,2}\Delta^{3,4}_{3,4} + \frac{1}{2}\Delta^{1,2}_{3,4}\Delta^{3,4}_{1,2} \\
\Delta^{1,2}\left(\Delta^{3,4}\right)^2 \,\&\, \Delta_{1,2}\left(\Delta_{3,4}\right)^2 &\rightarrow \frac{1}{3}\Delta^{1,2}_{1,2}\left(\Delta^{3,4}_{3,4}\right)^2 + \frac{2}{3}\Delta^{1,2}_{3,4}\Delta^{3,4}_{1,2}\Delta^{3,4}_{3,4}
\end{aligned} \tag{57}$$

The weighting in the sums occur because each side was computed independently, and thus some factors have been accounted for multiple times.

To see the source of these weights for an explicit example, consider the two operators $L_{3,2}$ and $R_{4,2}$ acting on the minor $\left(\Delta^{1,2}_{1,2}\right)^2$. Since the operators

commute, we can apply then in either order. The correct answer is

$$\begin{aligned} R_{4,2}L_{3,2}\left(\Delta_{1,2}^{1,2}\right)^2 &= R_{4,2}\left(2\Delta_{1,2}^{1,2}\Delta_{1,2}^{1,3}\right) \\ &= 2\left(\Delta_{1,4}^{1,2}\Delta_{1,2}^{1,3}+\Delta_{1,2}^{1,2}\Delta_{1,4}^{1,3}\right) \\ &= 2\Delta_{1,4}^{1,2}\Delta_{1,2}^{1,3}+2\Delta_{1,2}^{1,2}\Delta_{1,4}^{1,3}. \end{aligned} \tag{58}$$

The computer would compute the two sides as:

$$\begin{aligned} L_{3,2}\left(\Delta^{1,2}\right)^2 &= 2\Delta^{1,2}\Delta^{1,3}, \\ R_{4,2}\left(\Delta_{1,2}\right)^2 &= 2\Delta_{1,2}\Delta_{1,3}. \end{aligned} \tag{59}$$

Just summing the combinations (without the factors of $\frac{1}{2}$) would yield $4\Delta_{1,4}^{1,2}\Delta_{1,2}^{1,3}$ $+ 4\Delta_{1,2}^{1,2}\Delta_{1,4}^{1,3}$, which would be wrong. However, with the weighted sum, the combinations become

$$\begin{aligned} M\Delta^{(r,c)}\vec{h}^{(M)} &= [2]\left[\frac{1}{2}\Delta_{1,4}^{1,2}\Delta_{1,2}^{1,3}+\frac{1}{2}\Delta_{1,2}^{1,2}\Delta_{1,4}^{1,3}\right][2] \\ &= 2\Delta_{1,4}^{1,2}\Delta_{1,2}^{1,3}+2\Delta_{1,2}^{1,2}\Delta_{1,4}^{1,3}. \end{aligned} \tag{60}$$

Thus the factors account for any double counting contained in the coefficients.

Let h_i be a tableau corresponding to an element in $(m)_i$. The corresponding polynomial in $V^{(m)_i}$ can be written simply as $\vec{h}^{(i)}\cdot\vec{\Delta}^{(c,i)}$, where (i) denotes which $V^{(m)_i}$ space, and the (c) denotes reference to column indices. The values of $\vec{h}^{(i)}$ and $\vec{\Delta}^{(c,i)}$ for the space $V^{(m)_i}$ are computed similarly to the values $\vec{h}^{(M)}$ and $\vec{\Delta}^{(c)}$ for the space $V^{(M)}$. In this case, the rows remain the highest weight, and so the combinations discussed above for the $V^{(M)}$ space is avoided. Thus in this concrete representation, the Clebsch-Gordan coefficient of Equation (30) becomes, in inner product notation,

$$\frac{\left(P^{-1}M\Delta^{(r,c)}\vec{h}^{(M)},\prod_{i=1}^{r}\vec{h}^{(i)}\cdot\vec{\Delta}^{(c,i)}\right)}{\|P^{-1}M\Delta^{(r,c)}\vec{h}^{(M)}\|\|\prod_{i=1}^{r}\vec{h}^{(i)}\cdot\vec{\Delta}^{(c,i)}\|}, \tag{61}$$

with the norm defined as $\|f\|^2=(f,f)$.

This is as far as the spanning sets $\Delta^{(r,c)}$ and $\vec{\Delta}^{(c,i)}$ are useful. At this point one must finally break the minors of determinants into components to carry out the inner product. Even with advances in symbolic manipulation programs, directly carrying out Equation (61) can be incredibly complicated and computer intensive.

To overcome this difficulty, one now switches to another spanning set to isolate the complications. Note that each product of minors of determinants are linear combinations of products of the elements, in this case complex variables $z_{i,j}$. Thus one can enumerate all possible products of the $z_{i,j}$ as a new basis, denoted $\overrightarrow{(\pi \mathrm{Z})}$, and each element of $\Delta^{(r,c)}$ and $\vec{\Delta}^{(c,i)}$ can be written as a vector of coefficients dotted with $\overrightarrow{(\pi \mathrm{Z})}$. Computationally, it most practical to carry out the sums over elements of $\vec{h}^{(M)}$ and $\vec{h}^{(M)}$ in Equation (61) simultaneous with switching spanning sets. Thus we can define a matrix A such that

$\Delta^{(r,c)}_{i,j} h^{(M)}_j = A_{i,k}(\pi Z)_k$, and a vector $\vec{a}$ such that $\vec{a}\cdot\ \overrightarrow{(\pi \mathrm{Z})} = \prod_{i=1}^{r} \vec{h}^{(i)} \cdot \vec{\Delta}^{(c,i)}$. Thus the numerator of Equation (61) becomes

$$\left(P^{-1} M A\ \overrightarrow{(\pi \mathrm{Z})}, \vec{a}\cdot\ \overrightarrow{(\pi \mathrm{Z})} \right). \tag{62}$$

In index notation this becomes simply $(P^{-1}MA)_{i,j} a_k\, ((\pi Z)_j, (\pi Z)_k)$. Define the matrix D such that $D_{i,j} = ((\pi Z)_j, (\pi Z)_k)$. Using the inner product defined in Equation (11), one readily observes that D is a diagonal matrix, with the diagonal elements being simply the product of the factorials of the exponents of the $z_{i,j}$. The final computation becomes

$$\frac{\left(P^{-1}MADa\right)_i}{\sqrt{(P^{-1}MADA^T M^T P^{-T})_{i,i}}\sqrt{a^T D a}}, \tag{63}$$

where i corresponds an enumeration of the labels (η) corresponding to a chosen coupling scheme. Thus the only portion of the computation that is very intensive computationally is converting between the spanning set in terms of products of minors of determinants and the spanning set in terms of products of the variables. The complexity is comparable to computing $\overrightarrow{(\pi \mathrm{L})}\ \vec{\Delta}^{(r)}$ in the Racah coefficient calculation. Once the Clebsch-Gordan procedure has switched to the new basis $\overrightarrow{(\pi \mathrm{Z})}$, it can easily read off the norms of $\overrightarrow{(\pi \mathrm{Z})}$ and simple matrix multiplication yields the final result.

6 Examples

6.1 SU(2) 4-fold

The first example is chosen to be simple enough to explicitly show all the steps of the procedure, and yet complicated enough to show the power of the procedure. Consider combining four spin-$\frac{1}{2}$ objects. The total spin can then be any integer value between 2 and 0. Consider the multiplicity of the spin 1 irrep in the direct sum basis. In the language of these procedures, spin-$\frac{1}{2}$ objects correspond to the $SU(2)$ irrep (1,0), and the spin 1 objects correspond to the (2,0) irrep. The tensor product space is then $V^{(1,0)} \otimes V^{(1,0)} \otimes V^{(1,0)} \otimes V^{(1,0)}$. We choose $(M) = (3, 1, 0, 0)$ since $(3, 1)$ is equivalent to $(2, 0)$ and sums to the correct value. The highest weight vector is then

$$f_{\max} = \left(\Delta^1_1\right)^2 \Delta^{1,2}_{1,2} \tag{64}$$

There are three tableau:

$$\begin{pmatrix} 3 & & 1 & & 0 & & 0 \\ & 2 & & 1 & & 0 & \\ & & 2 & & 0 & & \\ & & & 1 & & & \end{pmatrix}, \begin{pmatrix} 3 & & 1 & & 0 & & 0 \\ & 2 & & 1 & & 0 & \\ & & 1 & & 1 & & \\ & & & 1 & & & \end{pmatrix},$$
$$\begin{pmatrix} 3 & & 1 & & 0 & & 0 \\ & 3 & & 0 & & 0 & \\ & & 2 & & 0 & & \\ & & & 1 & & & \end{pmatrix}. \tag{65}$$

The operators from the tableau are:

$$\overrightarrow{(\pi \mathrm{L})} = \begin{bmatrix} L_{4,2}L_{3,1}L_{2,1} \\ L_{4,1}L_{3,2}L_{2,1} \\ L_{4,1}L_{3,1} \end{bmatrix}. \tag{66}$$

Applying these operators to $f_{\max}$ and writing the result as $\vec{\Phi} = \overrightarrow{(\pi \mathrm{L})}\, f_{\max} = M\vec{\Delta}$ yields

$$M\vec{\Delta} = \begin{bmatrix} 2 & 0 & 0 & 2 & 0 & 2 \\ -2 & 0 & 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} \Delta^1\Delta^2\Delta^{3,4} \\ \Delta^3\Delta^4\Delta^{1,2} \\ \Delta^2\Delta^4\Delta^{1,3} \\ \Delta^2\Delta^3\Delta^{1,4} \\ \Delta^1\Delta^4\Delta^{2,3} \\ \Delta^1\Delta^3\Delta^{2,4} \end{bmatrix}. \tag{67}$$

For $SU(2)$, there are no off-diagonal elements and thus no Borel condition to solve. The above equation can also be written out explicitly as

$$\begin{aligned} \Phi_1 &= 2\,\Delta^1\Delta^2\Delta^{3,4} + 2\,\Delta^2\Delta^3\Delta^{1,4} + 2\,\Delta^1\Delta^3\Delta^{2,4} \\ \Phi_2 &= -2\,\Delta^1\Delta^2\Delta^{3,4} + 2\,\Delta^2\Delta^{1,3}\Delta^4 + 2\,\Delta^1\Delta^{2,3}\Delta^4 \\ \Phi_3 &= 2\,\Delta^{1,2}\Delta^3\Delta^4 - 2\,\Delta^1\Delta^{2,3}\Delta^4 - 2\,\Delta^1\Delta^3\Delta^{2,4}. \end{aligned} \tag{68}$$

For $SU(2)$, the Complete Coupling method works well and has integer eigenvalues. We first apply the operator coupling the set of spaces $\{1,2\}$ and then the operator coupling the set of spaces $\{1,2,3\}$, corresponding to the coupling scheme $(((1,2),3),4)$. This yields the following pairs of eigenvalues and P matrix:

$$\begin{bmatrix} (2,9) \\ (6,15) \\ (6,9) \end{bmatrix}, \begin{bmatrix} 0 & -3 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & -1/2 \end{bmatrix}. \tag{69}$$

Note that in this case it is not necessary to apply an operator coupling all of the spaces, $\{1,2,3,4\}$.

Alternately, we chose the coupling scheme of $((1,2),(3,4))$, and apply operators coupling the spaces $\{1,2\}$ and $\{3,4\}$. This yields the following eigenvalues and P matrix:

$$\begin{bmatrix} (2,6) \\ (6,6) \\ (6,2) \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}. \tag{70}$$

The Racah coefficients between the two coupling schemes are then computed to be:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/3\sqrt{3} & 1/3\sqrt{6} \\ 0 & 1/3\sqrt{6} & 1/3\sqrt{3} \end{bmatrix}. \tag{71}$$

In this case, the Gel'fand-Zetlin basis is nearly trivial, since the only elements of each spin-$\frac{1}{2}$ space are the highest and lowest weight vectors, which can be written directly without any of the above formalism. Thus examples of computing Clebsch-Gordan coefficients are deferred to more in-depth examples.

6.2 SU(3) 3-fold

Consider next the 3-fold tensor product of the eight dimensional representation of $(2,1,0)$, and consider the multiplicity of the same irrep in the tensor product. In this case, $(m)_1 = (m)_2 = (m)_3 = (2,1,0)$ and the top row of the tableau is $(4,3,2,0,0,0)$, since $(4,3,2)$ is equivalent to $(2,1,0)$ and has the proper weight. In this case, the highest weight vector of $V^{(M)} = V^{(4,3,2)}$ is

$$f_{\max} = \Delta_1^1 \Delta_{1,2}^{1,2} \left(\Delta_{1,2,3}^{1,2,3}\right)^2. \tag{72}$$

There are 28 tableau that yield 28 products of differential operators. The spanning set $\vec{\Delta}$ has dimension 168, and thus writing $\overrightarrow{(\pi \mathrm{L})}\, f_{\max} = \tilde{M}\vec{\Delta}$ implies the matrix $\tilde{M}$ has dimensions 28×168. The operators for the Borel condition are $L_{1,2}, L_{3,4}$, and $L_{5,6}$. There are eight linearly independent combinations such that

$$\begin{aligned} L_{1,2} B\, \overrightarrow{(\pi \mathrm{L})}\, f_{\max} &= 0, \\ L_{1,2} B\, \overrightarrow{(\pi \mathrm{L})}\, f_{\max} &= 0, \\ L_{1,2} B\, \overrightarrow{(\pi \mathrm{L})}\, f_{\max} &= 0, \end{aligned} \tag{73}$$

and thus the matrix B has dimension 8×28. We finally write $\vec{\Phi} = B\tilde{M}\vec{\Delta} = M\vec{\Delta}$, and the matrix M now has dimension 8×168.

We choose our first coupling scheme to be ((1,2),3) and choose to represent it using operators related to the Binary Coupling method of degree 3. The two generalized Casimir operators are then $[L]_{1,2}[L]_{2,2}[L]_{2,1}$ and $[L]_{1,3}[L]_{3,3}[L]_{3,1} +$

$[L]_{2,3}[L]_{3,3}[L]_{3,2}$. The eigenvalues are

$$\begin{bmatrix} 6 & 66 \\ 36 & 42 \\ 30 & 36 \\ \frac{39}{2}+\frac{3}{2}\sqrt{5} & \frac{105}{2}-\frac{3}{2}\sqrt{5} \\ \frac{39}{2}+\frac{3}{2}\sqrt{5} & \frac{105}{2}+\frac{3}{2}\sqrt{5} \\ \frac{39}{2}-\frac{3}{2}\sqrt{5} & \frac{105}{2}+\frac{3}{2}\sqrt{5} \\ \frac{39}{2}-\frac{3}{2}\sqrt{5} & \frac{105}{2}-\frac{3}{2}\sqrt{5} \\ 42 & 30 \end{bmatrix}. \tag{74}$$

We next choose the alternate coupling scheme of (1,(2,3)). The operators in this case are $[L]_{2,3}[L]_{3,3}[L]_{3,2}$ and $[L]_{2,1}[L]_{1,1}[L]_{1,2} + [L]_{3,1}[L]_{1,1}[L]_{1,3}$. The eigenvalues are

$$\begin{bmatrix} 36 & 42 \\ \frac{39}{2}+\frac{3}{2}\sqrt{5} & \frac{105}{2}-\frac{3}{2}\sqrt{5} \\ \frac{39}{2}+\frac{3}{2}\sqrt{5} & \frac{105}{2}+\frac{3}{2}\sqrt{5} \\ \frac{39}{2}-\frac{3}{2}\sqrt{5} & \frac{105}{2}+\frac{3}{2}\sqrt{5} \\ \frac{39}{2}-\frac{3}{2}\sqrt{5} & \frac{105}{2}-\frac{3}{2}\sqrt{5} \\ 30 & 36 \\ 42 & 30 \\ 6 & 66 \end{bmatrix}. \tag{75}$$

The Racah coefficients between these two coupling schemes are computed to be

$$\begin{bmatrix} \frac{\sqrt{10}}{8} & \frac{\sqrt{2}}{4} & 0 & \frac{\sqrt{2}}{4} & 0 & \frac{-\sqrt{10}}{8} & \frac{3\sqrt{3}}{8} & \frac{1}{8} \\ \frac{1}{4} & \frac{-\sqrt{5}}{10} & \frac{1}{2}+\frac{\sqrt{5}}{10} & \frac{\sqrt{5}}{10} & \frac{1}{2}-\frac{\sqrt{5}}{10} & \frac{-1}{4} & \frac{-\sqrt{30}}{40} & \frac{\sqrt{10}}{8} \\ \frac{1}{4} & \frac{-\sqrt{5}}{10} & \frac{-1}{2}+\frac{\sqrt{5}}{10} & \frac{\sqrt{5}}{10} & \frac{-1}{2}-\frac{\sqrt{5}}{10} & \frac{-1}{4} & \frac{-\sqrt{30}}{40} & \frac{\sqrt{10}}{8} \\ \frac{-1}{2}+\frac{\sqrt{5}}{10} & \frac{1}{5} & \frac{-1}{5} & \frac{-1}{5} & \frac{1}{5} & -\frac{1}{2}-\frac{\sqrt{5}}{10} & \frac{-\sqrt{6}}{5} & 0 \\ \frac{-\sqrt{5}}{10} & \frac{4}{5} & \frac{1}{5} & \frac{1}{5} & \frac{-1}{5} & \frac{\sqrt{5}}{10} & \frac{-\sqrt{6}}{20} & \frac{\sqrt{2}}{4} \\ \frac{1}{2}+\frac{\sqrt{5}}{10} & \frac{1}{5} & \frac{-1}{5} & \frac{-1}{5} & \frac{1}{5} & \frac{1}{2}-\frac{\sqrt{5}}{10} & \frac{-\sqrt{6}}{5} & 0 \\ \frac{-\sqrt{5}}{10} & \frac{-1}{5} & \frac{1}{5} & \frac{-4}{5} & \frac{-1}{5} & \frac{\sqrt{5}}{10} & \frac{-\sqrt{6}}{20} & \frac{\sqrt{2}}{4} \\ \frac{-\sqrt{30}}{40} & \frac{-\sqrt{6}}{20} & \frac{-\sqrt{6}}{5} & \frac{\sqrt{6}}{20} & \frac{\sqrt{6}}{5} & \frac{\sqrt{30}}{40} & \frac{7}{40} & \frac{3\sqrt{3}}{8} \end{bmatrix},$$

where the i, jth element corresponds to the overlap between elements labeled by the ith label in Equation (74) and the jth label in Equation (75).

To compute a Clebsch-Gordan coefficient, we need to first compute basis

elements. First, consider the element of $V^{(4,3,2)}$ labeled by

$$h^{(M)} = \begin{pmatrix} 4 & & 3 & & 2 \\ & 4 & & 2 & \\ & & 3 & & \end{pmatrix}, \tag{76}$$

equivalent to the tableau of (2,1,0)

$$\begin{pmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{pmatrix}. \tag{77}$$

The weight of the tableau $h^{(M)}$ is $(3,3,3)$. There is only one other tableau with this weight, namely

$$\begin{pmatrix} 4 & & 3 & & 2 \\ & 3 & & 3 & \\ & & 3 & & \end{pmatrix}. \tag{78}$$

Thus $\overrightarrow{(\pi \mathrm{R})}$ has two elements, each corresponding to operators from tableau of the same weight as the given tableau, one of these being the given tableau itself. We can then apply $\overrightarrow{(\pi \mathrm{R})}$ to $f_{\max}$ and write the results as $\overrightarrow{(\pi \mathrm{R})}\ f_{\max} = \tilde{M}^{(c)} \vec{\Delta}^{(c)}$. In this case,

$$\begin{aligned} \overrightarrow{(\pi \mathrm{R})}\ f_{\max} &= \begin{bmatrix} L_{3,2}L_{2,1} \\ L_{3,1} \end{bmatrix} \left(\Delta_1 \Delta_{1,2} \left(\Delta_{1,2,3} \right)^2 \right) = \\ &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \Delta_3 \Delta_{1,2} \left(\Delta_{1,2,3} \right)^2 \\ \Delta_2 \Delta_{1,3} \left(\Delta_{1,2,3} \right)^2 \\ \Delta_1 \Delta_{2,3} \left(\Delta_{1,2,3} \right)^2 \end{bmatrix}. \end{aligned} \tag{79}$$

The only difference between the two tableau of the same weight is the middle row, which contains two elements. Thus the quadratic Casimir over the first two variables is sufficient to break the degeneracy. The operator is

$$R_{1,1}R_{1,1} + R_{1,2}R_{2,1} + R_{2,1}R_{1,2} + R_{2,2}R_{2,2} \tag{80}$$

and the eigenvalues and matrix P' are:

$$\begin{bmatrix} 22 \\ 18 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix} \tag{81}$$

A known formula [10] associates the eigenvalue of 22 with the row $[4,2]$, and associates the first eigenvector with the tableau $h^{(M)}$. Combining the proper eigenvector of P'^{-1} with $\tilde{M}^{(c)}$ allows us to write the result as:

$$\vec{h}^{(M)} \cdot \vec{\Delta}^{(c)} = \begin{bmatrix} 0 & -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} \Delta_3 \Delta_{1,2} \left(\Delta_{1,2,3} \right)^2 \\ \Delta_2 \Delta_{1,3} \left(\Delta_{1,2,3} \right)^2 \\ \Delta_1 \Delta_{2,3} \left(\Delta_{1,2,3} \right)^2 \end{bmatrix}. \tag{82}$$

The basis elements in the tensor product basis are computed similarly.

The next step is to combine $\vec{\Delta}^{(r)}$ with $\vec{\Delta}^{(c)}$ in the direct sum basis, and similarly in the tensor product basis. Both sides are then written in terms of the new spanning set $\overrightarrow{(\pi \mathrm{Z})}$, which in this case has dimension 417. Let us choose the following basis elements:

$$h^{(1)} = \begin{pmatrix} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 2 & & \end{pmatrix}, \quad h^{(2)} = \begin{pmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{pmatrix},$$

$$h^{(3)} = \begin{pmatrix} 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{pmatrix}, \tag{83}$$

We shall then compute the Clebsch-Gordan coefficient in direct sum basis corresponding to the coupling scheme $((1,2),3)$, with the above choice of basis elements $h^{(M)}, h^{(1)}, h^{(2)}, h^{(3)}$. (Equations (30), (30), (61), and (63)). The values are:

$$\frac{1}{77760} \begin{bmatrix} 0 \\ 0 \\ -2 \\ -1-\sqrt{5} \\ 4-2\sqrt{5} \\ -1+\sqrt{5} \\ 4+2\sqrt{5} \\ -1 \end{bmatrix} \tag{84}$$

where the ith coefficient above corresponds element labeled by the ith set of eigenvalues in Equation (74). On an average desktop computer, the computation time of the above example is around two to three minutes total for all calculations.

7 Conclusion

Given any irrep (M) of $U(N)$ and any tensor product of irreps $(m)_1 \otimes (m)_2 \otimes \ldots \otimes (m)_r$ of $U(N)$, we have demonstrated procedures how to map elements from the representation space $V^{(M)}$ to the tensor product representations space $V^{(m)_1} \otimes V^{(m)_2} \otimes \ldots \otimes V^{(m)_r}$. The number of linearly independent maps yields the multiplicity. Furthermore, we have introduced generalized Casimir operators. Mutually commuting sets of generalized Casimir operators yield orthogonal bases among the multiplicity, and eigenvalues of these operators yield labels for the multiplicity. The relation between coupling schemes and generalized Casimir operators has been discussed. In particular, the author has presented his "Binary Coupling Method" of choosing generalized Casimir operators related to coupling schemes. Methods of computing Racah coefficients between

different mutually commuting sets of generalized Casimir operators have been presented, and although the computations began by using the highest weight element of $V^{(M)}$, the result is shown to be independent of this choice.

Procedures have also been provided to compute a representation of the Gel'fand-Zetlin basis for any $U(N)$ irrep. This follows very similarly to the procedure to map elements of $V^{(M)}$ into the tensor product space. Having a concrete realization of the Gel'fand-Zetlin basis, and having the multiplicity labeled by a set of commuting generalized Casimir operators, Clebsch-Gordan coefficients can then be easily computed by using the differential inner product. Thus both the multiplicity labeling problems and the general $U(N)$ basis problems have been solved, and both Racah and Clebsch-Gordan coefficients can be computed for any r-fold tensor product of $U(N)$. The author has programmed all of these procedures into Maple, and has optimized the program so that 3-fold $SU(3)$ examples can be computed within a few minutes, and 4-fold $SU(3)$ examples in a matter of hours. The author has also written online documentation explaining how to use the procedures, including several example worksheets. The program and documentation is available through a link at

`http://www.physics.uiowa.edu/~wklink`.

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