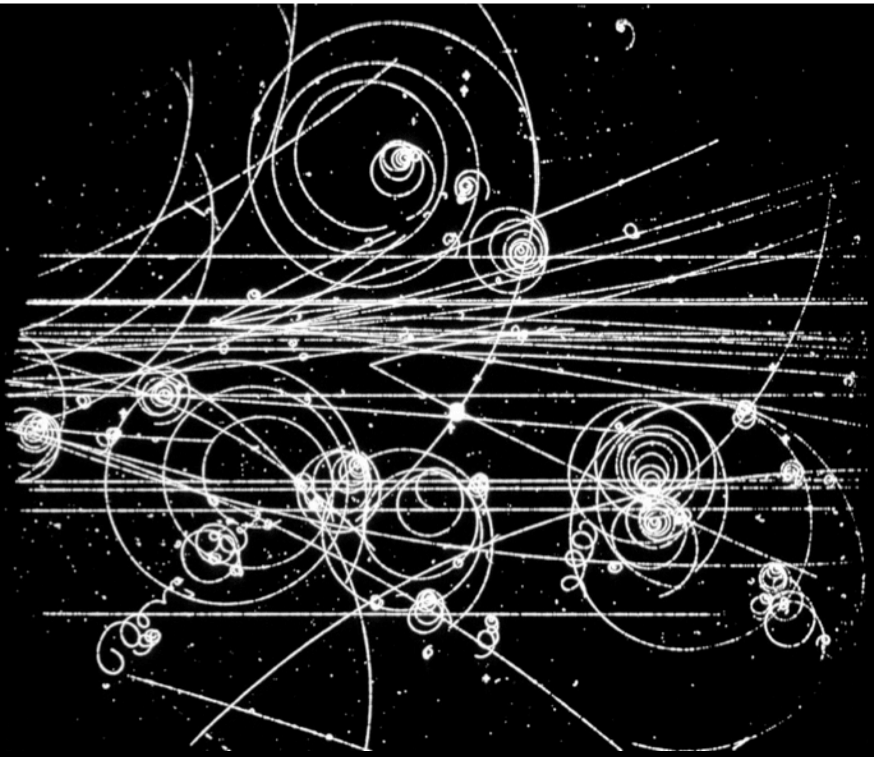
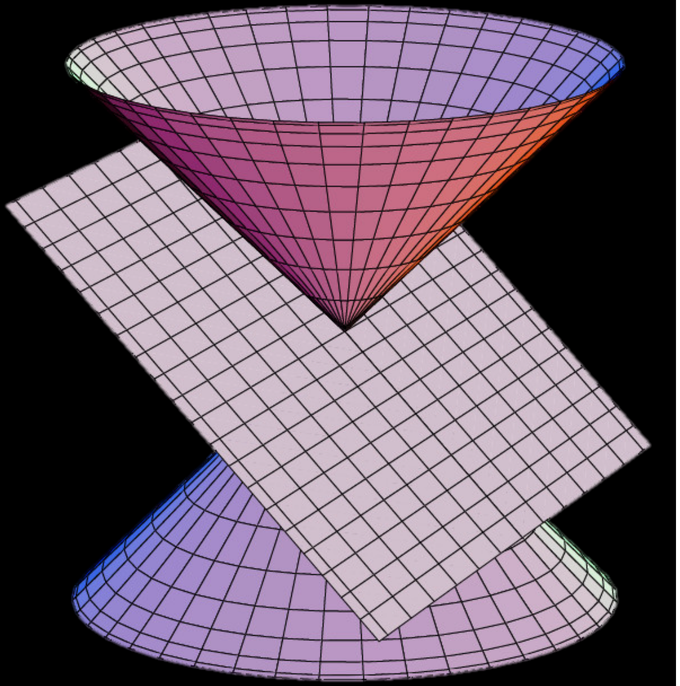
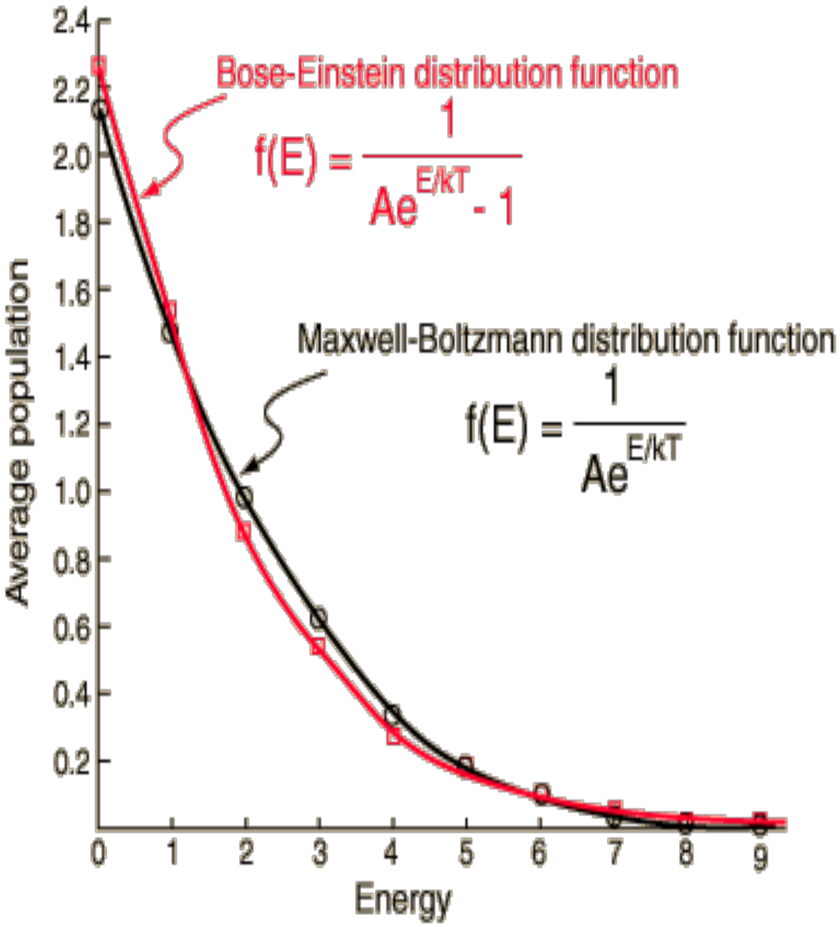


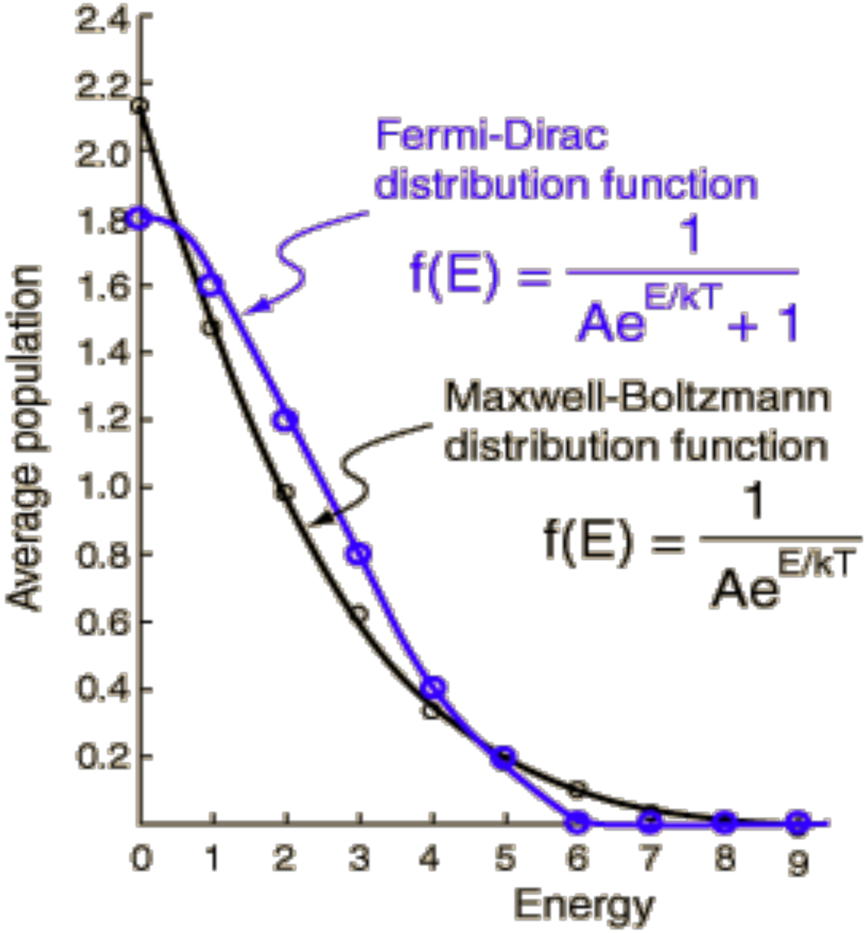


Modern Physics (Phys. IV): 2704

Professor Jasper Halekas
Van Allen 70
MWF 12:30-1:20 Lecture

Classical/Quantum Energy Distribution Functions





Degeneracy/Density of States

- d_n is the degeneracy (or maximum occupancy) of the energy level E_n
  - E.g. in the hydrogen atom the electron subshell $n = 2$, $l = 1$ has degeneracy $d_n = 6$
- g(E) is the density of states per unit volume
  - V g(E) is the continuous equivalent of d_n

Discrete Energy Distribution

$$N_n = d_n P_n$$

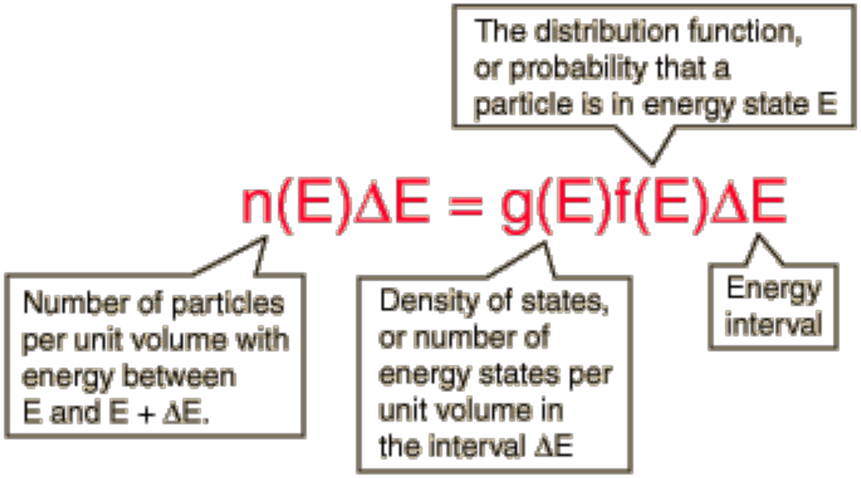
d_n = degeneracy of level n
P_n = probability of energy n

$$N = \sum_n N_n = \sum_n d_n P_n$$

$$\langle E \rangle = \frac{\sum_n N_n E_n}{N} = \frac{\sum_n d_n P_n E_n}{\sum_n d_n P_n}$$

Populated States

The number of populated states per unit volume n(E) is proportional to the product of the density of states g(E) and the energy distribution function f(E)



The distribution function, or probability that a particle is in energy state E

$$n(E)\Delta E = g(E)f(E)\Delta E$$

Number of particles per unit volume with energy between E and E + ΔE.

Density of states, or number of energy states per unit volume in the interval ΔE

Energy interval

Concept Check

- What is the correct normalization condition for N particles?

- Integral(V g(E) f(E) dE) = 1
- Integral(g(E) f(E) dE) = 1
- Integral(V g(E) f(E) dE) = N
- Integral(f(E) dE) = N

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Continuous Energy Distribution

$$dN = N(E)\,dE$$

$$= V\,g(E)\,f(E)\,dE$$

V = volume

$g(E)$ = density of states @ Energy E per volume

$f(E)$ = energy distribution function

= probability of energy E

$$N = \int dN = \int_0^\infty N(E)\,dE$$

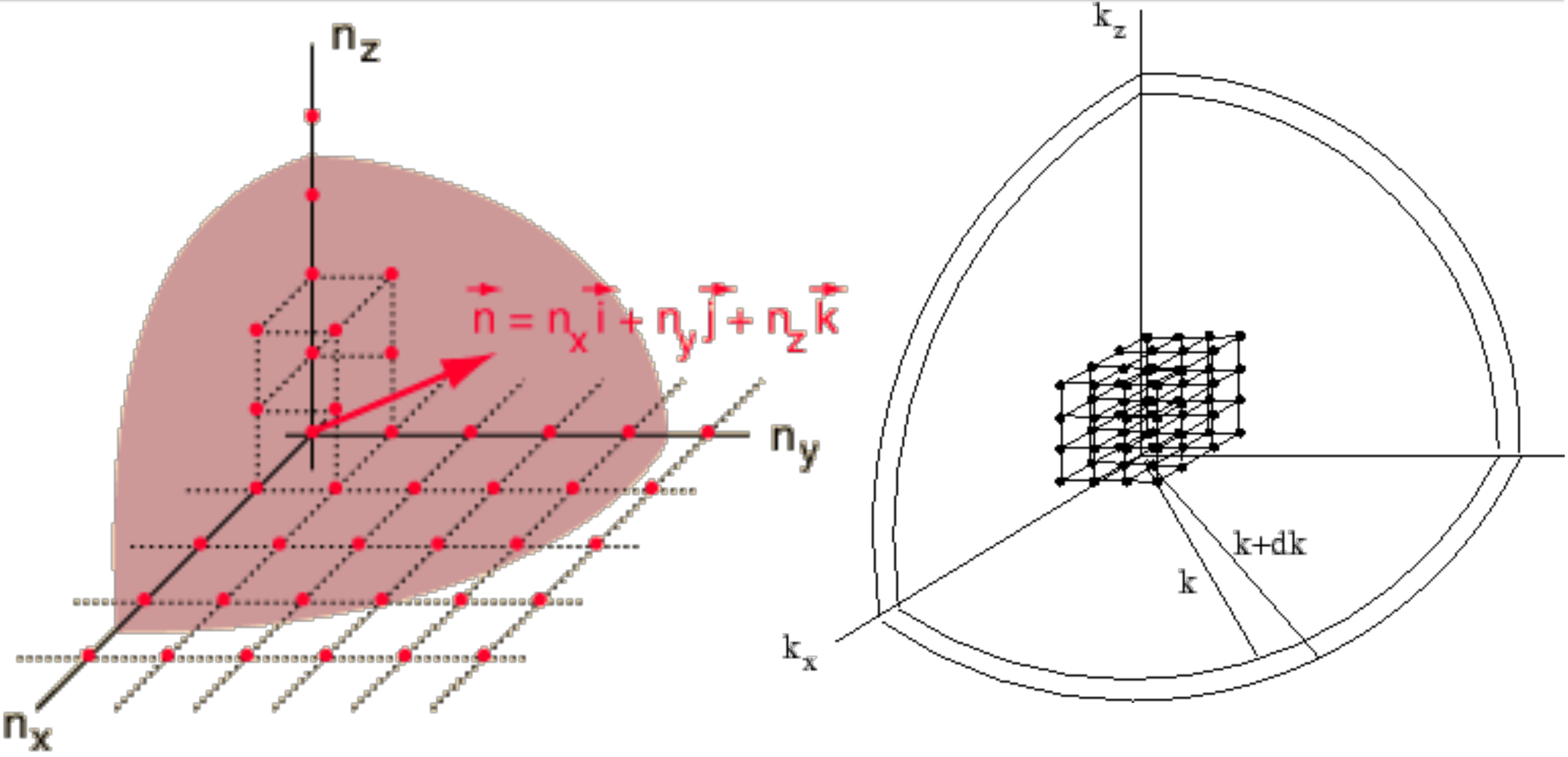
$$= \int_0^\infty V\,g(E)\,f(E)\,dE$$

$$\langle E \rangle = \frac{\int_0^\infty E\,N(E)\,dE}{N}$$

$$= \frac{V\int_0^\infty E\,g(E)\,f(E)\,dE}{V\int_0^\infty g(E)\,f(E)\,dE}$$

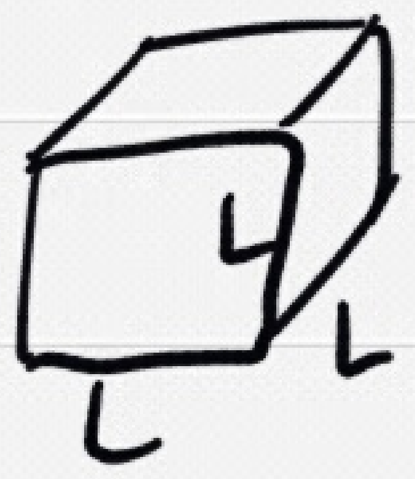
$$= \frac{\int_0^\infty E\,g(E)\,f(E)\,dE}{\int_0^\infty g(E)\,f(E)\,dE}$$

Density of States in a Gas in a Box



Density of States: Gas

Wave function for particle in box



$$\psi(x,y,z) = A \sin\left(\frac{n_x \pi x}{L}\right) \times \sin\left(\frac{n_y \pi y}{L}\right) \times \sin\left(\frac{n_z \pi z}{L}\right)$$

$$E = \frac{\hbar^2 \pi^2}{2mL^2}\left(n_x^2 + n_y^2 + n_z^2\right)$$
$$= \frac{\hbar^2}{2m}\left(k_x^2 + k_y^2 + k_z^2\right)$$

- # of states from E to $E+dE$
  - A shell in n_x, n_y, n_z space w/ all positive

$g(n)\,dn$ proportional to

$$\underset{n_x,n_y,n_z>0}{\underset{\uparrow}{\tfrac{1}{8}}}\ \underset{\text{Area of shell}}{\underset{\uparrow}{4\pi n^2}}\ \underset{\text{thicness of shell}}{\underset{\nwarrow}{dn}}$$

- spin s particles $\Rightarrow$ $2s+1$ orientations

$$g(n)\,dn = \frac{1}{8}\,\frac{2s+1}{V}\,4\pi n^2\,dn$$

$$g(E)\,dE = g(n)\,dn$$

$$g(E) = g(n) \Big/ \left(dE/dn \right)$$

$$E = \frac{\hbar^2 \pi^2 n^2}{2mL^2} \iff n = \sqrt{\frac{2mL^2E}{\hbar^2\pi^2}}$$

$$dE/dn = \frac{\hbar^2\pi^2 n}{mL^2}$$

$$\Rightarrow g(E) = \frac{1}{8}\,\frac{2s+1}{V}\,4\pi n^2 \Big/ \left(\frac{\hbar^2\pi^2}{mL^2}\, n \right)$$

$$= \frac{1}{8}\,\frac{2s+1}{V}\,4\pi n \Big/ \left(\frac{\hbar^2\pi^2}{mL^2} \right)$$

$$= 1/8\,\frac{2s+1}{V}\cdot 4\pi \cdot \sqrt{\frac{2mL^2E}{\hbar^2\pi^2}} \Big/ \left(\frac{\hbar^2\pi^2}{mL^2} \right)$$

$$= 1/8\,\frac{2s+1}{V}\cdot 4\pi\,\frac{\sqrt{2m}\,L\sqrt{E}}{\hbar^2\pi}\cdot\frac{mL^2}{\hbar^2\pi^2}$$

$$= \boxed{\frac{2s+1}{\sqrt{2}}\,\frac{m^{3/2}}{\pi^2\hbar^3}\,\sqrt{E}}$$

Note L^3 cancels V

Density of states: Photons

Same derivation, but

$$2s+1 \rightarrow 2 \quad (RH \text{ or } LH)$$

$$E = h\nu = \hbar\omega$$
$$= \hbar k c$$
$$= \frac{\hbar c \pi n}{L} \iff n = \frac{EL}{\hbar \pi c}$$

$$dE/dn = \hbar c \pi / L$$

$$g(n) = 2 \cdot \frac{1}{8} \cdot 4\pi n^2 \cdot 1/V$$

$$g(E) = g(n) / dE/dn$$
$$= \frac{\pi n^2}{V} \cdot \frac{L}{\hbar c \pi}$$
$$= \frac{\pi}{V} \frac{E^2 L^2}{\hbar^2 \pi^2 c^2} \cdot \frac{L}{\hbar c \pi}$$
$$= \boxed{\frac{E^2}{\pi^2 \hbar^3 c^3}}$$

- Fills hole in our derivation of Planck blackbody formula

$\boxed{\text{Maxwell - Boltzmann Distribution}}$

$$f(E) = \frac{1}{A} e^{-E/kT}$$

$$g(E) = \frac{2s+1}{\sqrt{2}} \frac{m^{3/2}}{\pi^2 \hbar^3} \sqrt{E}$$

$$N(E) = V g(E) f(E)$$

$$= \frac{1}{A} \cdot \frac{2s+1}{\sqrt{2}} \frac{m^{3/2}}{\pi^2 \hbar^3} \cdot V \cdot \sqrt{E} e^{-E/kT}$$

$$\int_0^\infty N(E)\, dE = N$$

$$\int_0^\infty \sqrt{E} e^{-E/kT} dE = \frac{\sqrt{\pi}}{2} (kT)^{3/2}$$

$$\Rightarrow \frac{1}{A} \frac{2s+1}{\sqrt{2}} \frac{m^{3/2}}{\pi^2 \hbar^3} \cdot V \cdot \frac{\sqrt{\pi}}{2} (kT)^{3/2} = N$$

$$\Rightarrow \boxed{N(E) = \frac{2N}{\sqrt{\pi} (kT)^{3/2}} \sqrt{E} e^{-E/kT}}$$

$\boxed{\text{M - B speed distribution}}$

$$N(v) = N(E) \, dE/dv = N(E) \cdot mv$$

$$= \frac{2N}{\sqrt{\pi} (kT)^{3/2}} \cdot \sqrt{\tfrac{1}{2} m v^2} \cdot mv \cdot e^{-\frac{1}{2} m v^2 / kT}$$

$$= \boxed{N \sqrt{\frac{2}{\pi}} \left(\frac{m}{kT}\right)^{3/2} \cdot v^2 e^{-mv^2/2kT}}$$

M-B velocity distribution

$$\int_0^\infty N(v)\,dv = \iiint N(v_x, v_y, v_z)\,dv_x\,dv_y\,dv_z$$

$$= \iiint N(v, \theta_v, \varphi_v)\cdot v^2 dv \sin\theta\, d\theta\, d\varphi$$

$$= 4\pi \int_0^\infty N(v, \theta_v, \varphi_v)\cdot v^2\, dv$$

$$\Rightarrow N(v, \theta_v, \varphi_v) = N(v_x, v_y, v_z)$$

$$= N(v)\cdot\frac{1}{4\pi v^2}$$

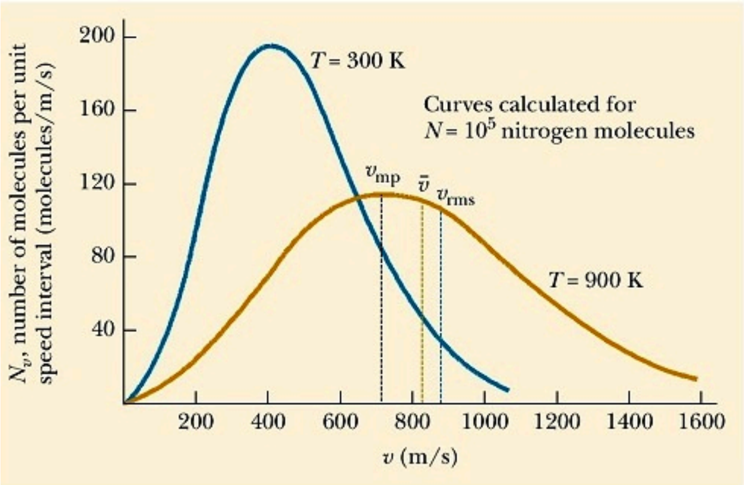
$$= N \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{1}{4\pi} \cdot \left(\frac{m}{kT}\right)^{3/2} \cdot v^2 e^{-mv^2/2kT} \cdot \frac{1}{v^2}$$

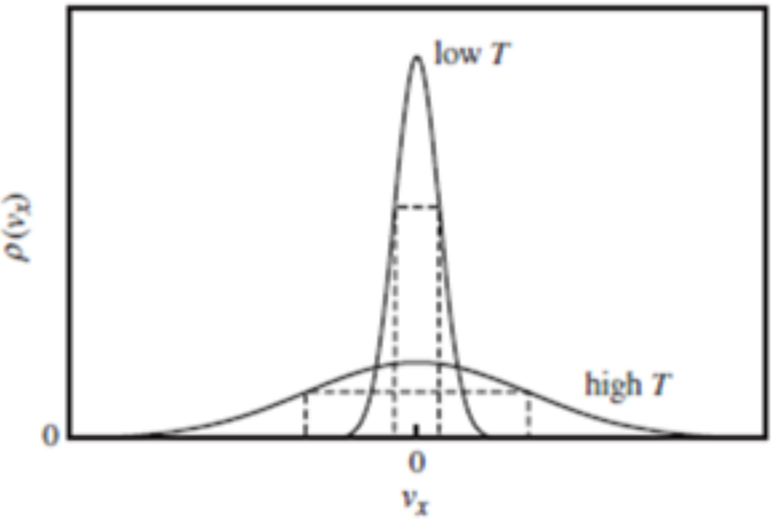
$$= \boxed{N\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-m(v_x^2+v_y^2+v_z^2)/2kT}}$$

1-d equivalent

$$\boxed{N(v_x) = N\sqrt{\frac{m}{2\pi kT}}\; e^{-mv_x^2/2kT}}$$

Maxwell-Boltzmann Distribution





Doppler Broadening

put v_x = "line-of-sight" velocity

Doppler shift

$$f = \sqrt{\frac{1 - v_x/c}{1 + v_x/c}}\; f_0$$

$$\sim f_0 \left(1 - v_x/c\right) \qquad |v_x| << c$$

$$\Rightarrow v_x = c\left(1 - f/f_0\right)$$

$$N(f) = N(v_x) / \left| df/dv_x \right|$$

$$= N(v_x) / (f_0/c)$$

$$= N(v_x) \cdot c/f_0$$

$$= \frac{Nc}{f_0} \sqrt{\frac{m}{2\pi kT}}\; e^{-mc^2\left(1 - f/f_0\right)^2/2kT}$$

$$\boxed{\text{FWHM} \quad \Delta f = 2 f_0 \sqrt{(2\ln 2)\, kT/mc^2}}$$

Doppler Broadening