Euclidean formulation of relativistic quantum mechanics

W. N. Polyzou

The University of Iowa

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Motivation

Happy Birthday Walter!

W. Glöckle and L. Müeller

(W. Glöckle and L. Müller, Phys. Rev. C23, 1183 (1981),

A. Kruger, Walter Glöckle, Phys. Rev. C60, 024004(1999))

Relativistic quantum mechanics from (Euclidean) field theory

Collaborators

Iowa Students

Phil Kopp, Victor Wessels

Supported by U.S. Department of Energy

Input:

Assume we are given a model of a reflection-positive Euclidean Green function or generating functional.

Question:

Can we use this as a practical framework for computations in relativistic quantum mechanics?

(Without using analytic continuation!)

Definitions:

Euclidean generating functional or Green functions:

$$Z[f] := \frac{\int D_e[\phi] e^{-\mathcal{A}[\phi] + i\phi(f)}}{\int D_e[\phi] e^{-\mathcal{A}[\phi]}} = \sum_n \frac{(i)^n}{n!} G_n \underbrace{(f, \cdots, f)}_{\text{n times}}$$

Positive Euclidean-time test functions

$$\mathcal{S}_+ := \{ f(\tau, \mathbf{x}) \in \mathcal{S} | f(\tau, \mathbf{x}) = 0, \quad \tau < 0 \}$$

Euclidean time reflection

$$\Theta f(\tau, \mathbf{x}) := f(-\tau, \mathbf{x})$$

Reconstruction of quantum mechanics

(Osterwalder and Schrader)

Vectors (wave functionals)

$$A[\phi] = \sum_{j=1}^{N_a} c_j e^{i\phi(f_j)} \qquad B[\phi] = \sum_{k=1}^{N_b} d_k e^{i\phi(g_k)}$$

$$c_j, d_k \in \mathbb{C} \qquad f_j, g_k \in \mathcal{S}_+ \qquad N_a, N_b < \infty$$

Physical Hilbert space inner product

$$\boxed{\langle A|B\rangle = \sum_{j=1}^{N_a} \sum_{k=1}^{N_b} c_j^* d_k Z[g_k - \Theta f_j]}$$

Properties of $Z[f]$

Reflection positivity

(Osterwalder-Schrader positivity)

$$\boxed{\langle A|A\rangle \geq 0}$$

$$M_{ij} = Z[f_i - \Theta f_j] \geq 0 \qquad \{f_1, \cdots, f_N\} \in \mathcal{S}_+$$

Cluster properties

$$g_{\mathbf{a}}(\tau, \mathbf{x}) := g(\tau, \mathbf{x} - \mathbf{a})$$

$$\boxed{\lim_{|\mathbf{a}| \to \infty} \left(Z[f + g_{\mathbf{a}}] - Z[f]Z[g] \right) \to 0}$$

$*$ Algebra

(used in the scattering asymptotic condition)

$$A[\phi]B[\phi] = \sum_{j=1}^{N_a} \sum_{k=1}^{N_b} c_j d_k e^{i\phi(g_k + f_j)} = \sum_{j=1}^{N_c} b_j e^{i\phi(h_j)} = C[\phi]$$

A reflection positive generating functional with time reflection is the vacuum functional on this algebra for the GNS construction of the physical Hilbert space.

Dynamics

$$g_\beta(\tau, \mathbf{x}) := g(\tau - \beta, \mathbf{x}) \qquad g \in \mathcal{S}_+ \qquad \beta > 0$$

$$\boxed{\langle A | e^{-\beta H} | B \rangle = \sum_{j=1}^{N_a} \sum_{k=1}^{N_b} c_j^* d_k Z[g_{\beta,k} - \Theta f_j]}$$

$$\langle A | H | B \rangle = -\frac{\partial}{\partial \beta} \left(\sum_{j=1}^{N_a} \sum_{k=1}^{N_b} c_j^* d_k Z[g_{\beta,k} - \Theta f_j] \right)_{\beta=0}$$

$$H = H^\dagger \geq 0$$

Lorentz boosts

$$g_{\eta,\hat{\mathbf{n}}}(\tau,\mathbf{x}) := g(\tau',\mathbf{x}') \qquad g \in \mathcal{S}_{\phi,+}$$

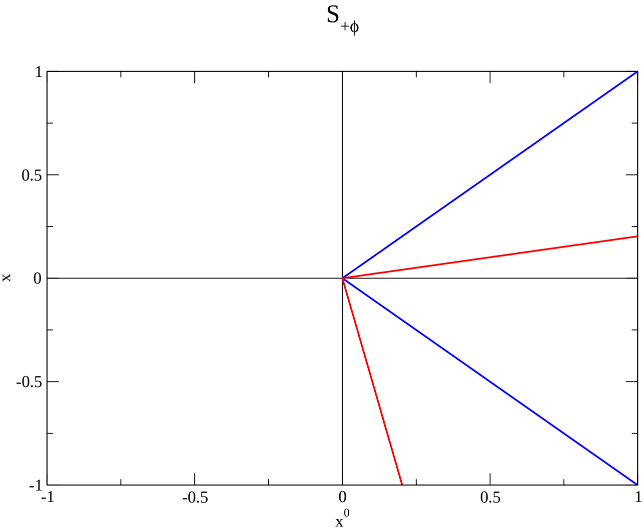
$$\tau' = \tau\cos(\eta) - x_{\hat{\mathbf{n}}}\sin(\eta) \qquad x'_{\hat{\mathbf{n}}} = x_{\hat{\mathbf{n}}}\cos(\eta) + \tau\sin(\eta)$$

$$\boxed{\langle A|e^{-\eta\hat{\mathbf{n}}\cdot\mathbf{K}}|B\rangle = \sum_{j=1}^{N_a}\sum_{k=1}^{N_b} c_j^* d_k Z[g_{k,\eta,\hat{\mathbf{n}}} - \Theta f_j]}$$

$$\langle A|\hat{\mathbf{n}}\cdot\mathbf{K}|B\rangle = -\frac{\partial}{\partial\eta}\left(\sum_{j=1}^{N_a}\sum_{k=1}^{N_b} c_j^* d_k Z[g_{k,\eta,\hat{\mathbf{n}}} - \Theta f_j]\right)_{\eta=0}$$

$$\mathbf{K} = \mathbf{K}^\dagger$$

$e^{-\mathbf{K}\eta\hat{\mathbf{n}}}$ symmetric local semigroup $\rightarrow$ $\mathbf{K} = \mathbf{K}^\dagger$



Rotations

$$g_{\boldsymbol{\eta}}(\tau, \mathbf{x}) := g(\tau, R(\boldsymbol{\eta})\mathbf{x}) \qquad g \in \mathcal{S}_+ > 0$$

$$\boxed{\langle A|e^{-i\mathbf{J}\cdot\boldsymbol{\eta}}|B\rangle = \sum_{j=1}^{N_a}\sum_{k=1}^{N_b} c_j^* d_k Z[g_{\boldsymbol{\eta},k} - \Theta f_j]}$$

$$\langle A|\mathbf{J}|B\rangle = i\frac{\partial}{\partial\boldsymbol{\eta}}\left(\sum_{j=1}^{N_a}\sum_{k=1}^{N_b} c_j^* d_k Z[g_{\boldsymbol{\eta},k} - \Theta f_j]\right)_{\boldsymbol{\eta}=0}$$

$$\mathbf{J} = \mathbf{J}^\dagger$$

Translations

$$g_{\mathbf{a}}(\tau, \mathbf{x}) := g(\tau, \mathbf{x} - \mathbf{a}) \qquad g \in \mathcal{S}_+ > 0$$

$$\boxed{\langle A | e^{i\mathbf{P}\cdot\mathbf{a}} | B \rangle = \sum_{j=1}^{N_a} \sum_{k=1}^{N_b} c_j^* d_k Z[g_{\mathbf{a},k} - \Theta f_j]}$$

$$\langle A | \mathbf{P} | B \rangle = -i \frac{\partial}{\partial \mathbf{a}} \left(\sum_{j=1}^{N_a} \sum_{k=1}^{N_b} c_j^* d_k Z[g_{\mathbf{a},k} - \Theta f_j] \right)_{\mathbf{a}=0}$$

$$\mathbf{P} = \mathbf{P}^\dagger$$

$$\{H, \mathbf{P}, \mathbf{J}, \mathbf{K}\}$$

Hermitian

Satisfy Poincaré commutation relations

$$\Downarrow$$

$$U(\Lambda, a) : \mathcal{H} \to \mathcal{H}$$

$$\Downarrow$$

Relativistic quantum theory

No analytic continuation used!

Given a reflection positive Euclidean Green function or generating function we have:

- **The physical Hilbert space scalar product,** $\langle A|B\rangle$
- **A dense set of normalizable vectors,** $A[\phi]$
- **A unitary representation,** $U(\Lambda, a)$**, of the Poincaré group on the physical Hilbert space.**

Comments:

While analytic continuation is not used, reflection positivity ensures the existence of an analytic continuation.

Momentum eigenstates

$$g_{\mathbf{a}}(\tau, \mathbf{x}) := g(\tau, \mathbf{x} - \mathbf{a}) \qquad g \in \mathcal{S}_+ > 0$$

$$\langle B | A_{\mathbf{p}} \rangle := \frac{1}{(2\pi)^{3/2}} \int d\mathbf{a} e^{-i\mathbf{p}\cdot\mathbf{a}} \sum_{j=1}^{N_a} \sum_{k=1}^{N_b} c_j^* d_k Z[g_{\mathbf{a},k} - \Theta f_j]$$

$$A_{\mathbf{p}}[\phi] := \int \frac{d\mathbf{a}}{(2\pi)^{3/2}} e^{-i\mathbf{p}\cdot\mathbf{a}} \sum_{j=1}^{N_a} d_k e^{i\phi(g_{\mathbf{a},k})}$$

$$A[\phi] = \int d\mathbf{p} A_{\mathbf{p}}[\phi] f(\mathbf{p})$$

Mass eigenstates

$$\langle A_{n,\mathbf{p}} | A_{m,\mathbf{p}'} \rangle = \delta_{nm} \delta(\mathbf{p} - \mathbf{p}')$$

$$\psi_{\lambda,\mathbf{p}}[\phi] = \sum_{n} A_{n,\mathbf{p}}[\phi] a_n(\mathbf{p})$$

$$\boxed{\sum_{m} \langle A_{n,\mathbf{p}} | (H^2 - \mathbf{P}^2) | A_{m,\mathbf{p}'} \rangle a_m(\mathbf{p}) = \delta_{nm} \delta(\mathbf{p} - \mathbf{p}') \lambda^2 a_n(\mathbf{p})}$$

Spin eigenstates

$$\psi_{\lambda,j,\mathbf{p},\mu}[\phi] =$$

$$\int U(R)\psi_{\lambda,R^{-1}\mathbf{p}}[\phi]D^{j*}_{\mu j}(R)dR$$

For $\lambda \in \sigma_{pp}(M)$ these create mass λ spin j single-particle states (irreducible representation spaces)

(spin is canonical spin)

Poincaré transformations (one-particle states)

$$\langle B|U(\Lambda, a)|\psi_{\lambda,j,\mathbf{p},\mu}\rangle =$$

$$\langle B|\psi_{\lambda,j,\mathbf{p}',\mu'}\rangle \sqrt{\frac{\omega_\lambda(\mathbf{p}')}{\omega_\lambda(\mathbf{p})}} e^{-i\omega_\lambda(\mathbf{p}')a^0+i\mathbf{p}'\cdot\mathbf{a}} D^j_{\mu'\mu}[\Lambda_c^{-1}(\frac{\mathbf{p}'}{\lambda})\Lambda\Lambda_c(\frac{\mathbf{p}}{\lambda})]$$

$$(\mathbf{p}')^j = \Lambda^j{}_0\omega_\lambda(\mathbf{p}) + \Lambda^j{}_k\mathbf{p}^k \qquad \omega_\lambda(\mathbf{p}) = \sqrt{\lambda^2+\mathbf{p}^2}$$

$\Lambda_c(\frac{\mathbf{p}}{\lambda}) :=$ **a rotationless boost.**

Scattering with Euclidean Green functions

Approximation 1: **Use sharply peaked (in momentum) normalizable states to approximate plane-wave on-shell transition matrix elements.**

$$\langle A|S|B\rangle = \delta_{ab}\langle A|B\rangle - 2\pi i\langle A|\delta(E_a - E_b)T^{ab}|B\rangle$$

$$\langle \mathbf{p}'_1, \mu'_1, \cdots, \mathbf{p}'_n, \mu'_n|T^{ab}|\mathbf{p}_1, \mu_1, \mathbf{p}_2, \mu_2\rangle \approx \frac{\langle A|S|B\rangle - \delta_{ab}\langle A|B\rangle}{2\pi i\langle A|\delta(E_a - E_b)|B\rangle}$$

Scattering injection operators

Approximation 2: **Construct** $\int \psi_{\lambda,j,\mathbf{p},\mu}[\phi] f(\mathbf{p},\mu) d\mathbf{p}$

Given a narrow wave packet, $f(\mathbf{p},\mu)$**, solve for the one-particle eigenstate.**

(Using two-Hilbert space formulation of Haag-Ruelle scattering gives wave operators as strong limits.)

$$\Phi[\phi] : \mathcal{H}_a \to \mathcal{H}$$

$$\Phi[\phi](\otimes f_i) := \sum \int \prod \psi_{\lambda_i,j_i,\mathbf{p}_i,\mu_i}[\phi] f_i(\mathbf{p}_i,\mu_i) d\mathbf{p}_i$$

Scattering in Euclidean space

Use time-dependent scattering to calculate S-matrix elements in normalizable states.

Use the Kato-Birman invariance principle to express S in terms of $e^{-\beta H}$, $H \rightarrow -e^{-\beta H}$.

$$\langle A|S|B\rangle = \lim_{t\to\infty} \langle \otimes f_{al} e^{-iH_a t} \Phi_a^\dagger | e^{2iHt} | \Phi_b e^{-iH_b t} \otimes f_{bl} \rangle$$

$$= \lim_{n\to\infty} \langle \otimes f_{al} e^{ine^{-\beta H_a}} \Phi_a^\dagger | e^{-2ine^{-\beta H}} | \Phi_b e^{ine^{-\beta H_b}} \otimes f_{bl} \rangle$$

$$\Phi_b[\phi](e^{ine^{-\beta H_a}} \otimes f_{bl}) :=$$

$$\sum\!\!\!\!\!\!\int [\prod \psi_{\lambda_l, j_l, \mathbf{p}_l, \mu_l}[\phi] f_{bl}(\mathbf{p}_l, \mu_l) d\mathbf{p}_l] e^{-ine^{-\beta(\sum_k \sqrt{\mathbf{p}_k^2 + \lambda_k^2})}}$$

Scattering in Euclidean space

Approximation 3: Replace $\lim_{n\to\infty}$ by a large fixed n.

$$\langle A|S|B\rangle \approx \langle A_s(n)|e^{-2ine^{-\beta H}}|B_s(n)\rangle$$

$$B_s(n)[\phi] := \Phi_b[\phi](e^{ine^{-\beta H_b}} \otimes f_{bk})$$

Approximation 4: Uniform polynomial approximation

$$e^{-2ine^{-\beta H}} \approx \sum c_m(n)(e^{-\beta H})^m$$

note $\sigma(e^{-\beta H}) \in [0,1]$ **(compact)**

$$e^{-2inx} \approx \sum c_m(n)x^m \qquad x \longrightarrow e^{-\beta H}$$

$$|e^{-2inx} - \sum c_m(n)x^m| < \epsilon(n) \qquad \forall x \in [0,1]$$

$$\Downarrow$$

$$|||[e^{-2ine^{-\beta H}} - \sum c_m(n)(e^{-\beta H})^m]||| < \epsilon(n)$$

Tchebichef + Gauss-Tchebichef quadrature ≈ best uniform approximation

$$f(x) \approx \frac{1}{2}c_0 T_0(x) + \sum_{k=1}^{N} c_k T_k(x)$$

$$c_j = \frac{2}{N+1}\sum_{k=1}^{N} f(\cos(\frac{2k-1}{N+1}\frac{\pi}{2}))\cos(j\frac{2k-1}{N+1}\frac{\pi}{2})$$

$$f(e^{-\beta H}) \approx \frac{1}{2}c_0 T_0(e^{-\beta H}) + \sum_{k=1}^{N} c_k T_k(e^{-\beta H})$$

$$f(x) = e^{-2inx}$$

$$|e^{-2inx} - P_N(x)| < 2\frac{n^{N+1}}{(N+1)!} \qquad x \in [-1,1]$$

Structure of the approximation

$$\langle \mathbf{p}'_1, \mu'_1, \cdots, \mathbf{p}'_n, \mu'_n | T^{ab} | \mathbf{p}_1, \mu_1, \mathbf{p}_2, \mu_2 \rangle \approx \frac{\langle A|S|B\rangle - \delta_{ab}\langle A|B\rangle}{2\pi i \langle A|\delta(E_a - E_b)|B\rangle}$$

For sufficiently large *n*

$$\langle A_s(n)|S|B_s(n)\rangle \approx \sum c_m(n) \langle A_s(n)|(e^{-\beta H})^m|B_s(n)\rangle$$

Observations

- [Given a model generating functional] the approximations are mathematical approximations with controlled errors. They are not truncations.
- Poincaré transformations of asymptotic states give transition-matrix elements in any frame.
- Model generating functionals (or Green functions) are the appropriate phenomenological input.
- Single-particle states and bound states are point eigenstates of the mass operator.

Approximations

- **Approximation 1:** This requires that the transition matrix elements are continuous functions of momenta. It is standard assumption in the derivation of time-independent scattering,
- **Approximation 2:** This assumes that the vectors $A[\phi]$ are dense. It is a strong limit in the wave operators and a weak limit in the S-matrix.
- **Approximation 3:** Is a strong limit in the wave operators and weak limit in the S matrix. It assumes that the Kato-Birman theorem holds.
- **Approximation 4:** This is a uniform limit for fixed n - it is easy to compute rigorous bounds.
- The order of the approximations is important: $(1) \to (2) \to (3) \to (4)$.

Lunch Menu

- $ **Euclidean invaraince**
- $ **Cluster properties**
- $$ **One particle states (**$\in \overline{A[\phi]}|0\rangle$**)**
- $$$ **Reflection positivity**

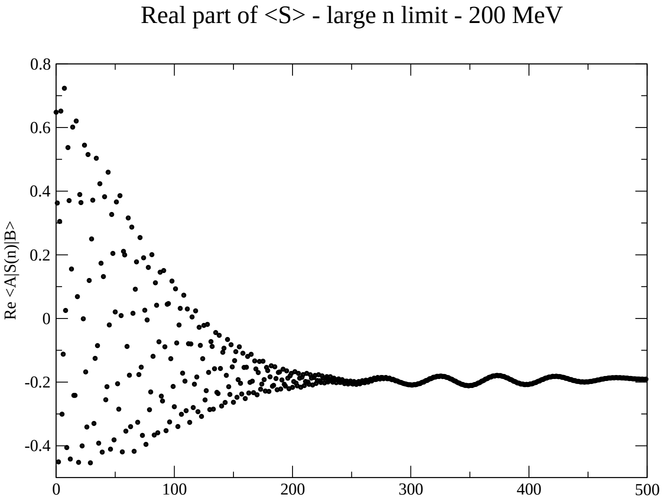
Test of method: non-relativistic separable potential

$$H = \frac{\mathbf{k}^2}{m} - |g\rangle\lambda\langle g|$$

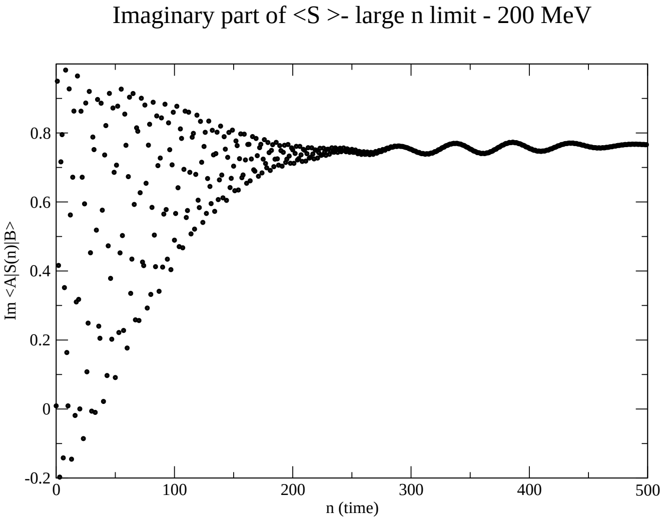
$$\langle \mathbf{k}|g\rangle = \frac{1}{m_\pi^2 + \mathbf{k}^2}$$

calculate $\langle k|T(k^+)|k\rangle$

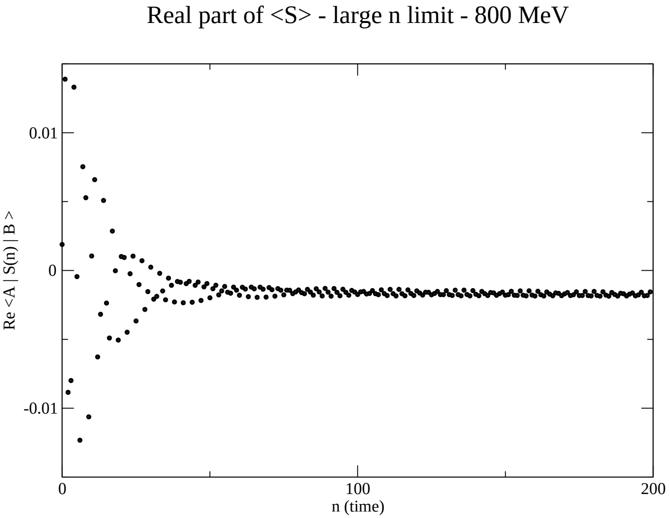
Approximation 3:



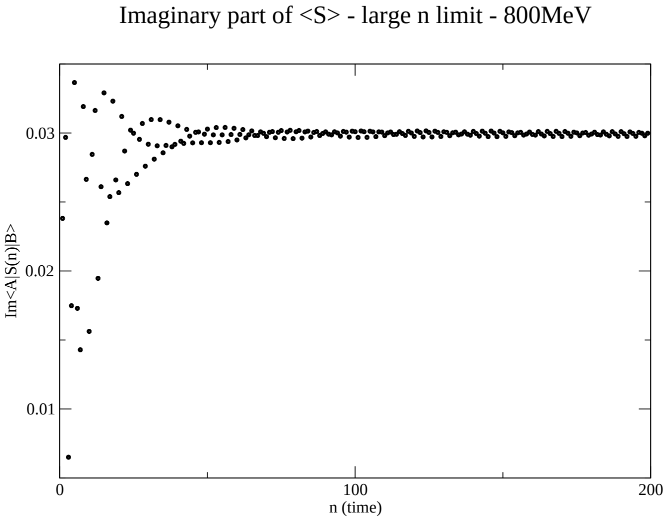
Approximation 3:



Approximation 3:



Approximation 3:

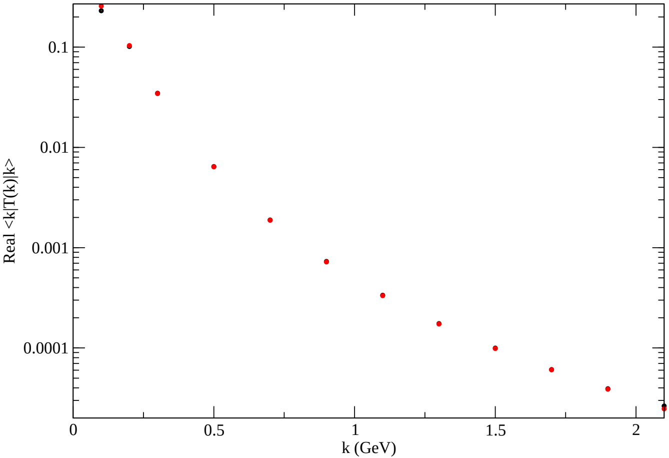


Approximation 4:

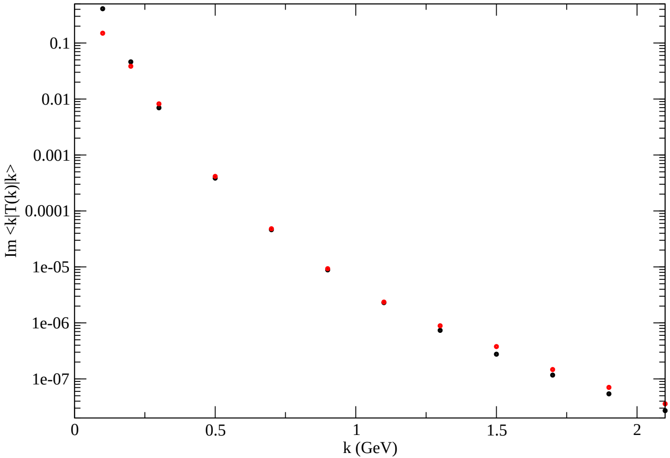
Degree 300 polynomial compared to e^{-inx}, $n = 220$

x	$\Delta \cos(nx)$	$\Delta \sin(nx)$
0	4.44089×10^{-16}	8.32667×10^{-15}
0.1	2.35367×10^{-14}	1.46966×10^{-14}
0.2	5.55112×10^{-16}	3.6797×10^{-14}
0.3	3.84137×10^{-14}	1.80689×10^{-14}
0.4	1.72085×10^{-14}	1.32672×10^{-14}
0.5	2.77556×10^{-15}	2.93793×10^{-14}
0.6	6.66134×10^{-16}	3.33344×10^{-14}
0.7	8.54872×10^{-15}	2.50355×10^{-14}
0.8	1.02141×10^{-14}	1.35447×10^{-14}
0.9	1.22125×10^{-15}	2.72282×10^{-14}
1	4.88498×10^{-15}	6.61415×10^{-14}

Real part of <k|T(k)|k> (exact - black, polynomial - red)



Im part of <k|T(k)|k> (exact - black, polynomial - red)



Conclusions - Outlook

- **Phenomenology based on model reflection-positive Euclidean Green functions can be used to formulate a relativistic quantum theory.**
- **Analytic continuation is not necessary.**
- **The Poincaré invariant S-matrix can be computed in any frame. Cluster properties are easily satisfied.**
- **Models can be motivated by field-theory based phenomenology.**
- **A test using an exactly solvable model suggests that scattering calculations are possible in this framework.**

Happy Birthday Walter!