

# From field theory to the Poincaré invariant three-body problem

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Research supported by:  
U.S. Department of Energy  
Office of Science

October 13, 2009

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# Special Relativity in Quantum Mechanics

E. P. Wigner - Ann. Math. 40,149(1939)

$$X, X' \quad \text{inertial} \quad (x - y)^2 = (x' - y')^2$$



$$x' = \Lambda x + a \quad \Lambda \eta \Lambda^t = \eta$$

**Poincaré group = symmetry group of quantum theory**

$$|\langle \phi | \psi \rangle|^2 = |\langle \phi' | \psi' \rangle|^2$$

$$|\psi'\rangle = U(\Lambda, \mathbf{a})|\psi\rangle$$

## Construction of $U(\Lambda, a)$

### Quantum field theory

#### vectors

$$|g\rangle = \phi(g)|0\rangle = \int \phi(x)g(x)d^4x|0\rangle$$

$$|\mathbf{g}\rangle = \sum_n c_n \phi(g_{nn}) \cdots \phi(g_{n1})|0\rangle$$

#### inner product

$$\langle f|g\rangle = \int f^*(x) d^4x \langle 0|\phi^\dagger(x)\phi(y)|0\rangle d^4y g(y) = \int f^*(x) d^4x W(x; y) d^4y g(y)$$

$$\langle \mathbf{f}|\mathbf{g}\rangle =$$

$$\sum_{mn} c_n^* d_m \int g_n^*(x_1, \dots, x_n) W(x_1 \cdots x_n; y_m \cdots y_1) f_m(y_1, \dots, y_m) d^{4n}x d^{4m}y$$

$$U(\Lambda, a)$$

$$|\mathbf{g}'\rangle = U(\Lambda, a)|\mathbf{g}\rangle$$

$$g_m(x_1 \cdots x_n) \rightarrow g'_m(x_1 \cdots x_n) = \prod S^t(\Lambda^{-1}) g_m(\Lambda^{-1}(x_1 - a), \cdots, \Lambda^{-1}(x_n - a))$$

$U(\Lambda, a)$  **unitary**



$$W(x_1 \cdots x_n; y_m \cdots y_1) = \prod S_{mk}^*(\Lambda^{-1}) W(\Lambda x_1 + a, \cdots, \Lambda y_1 + a) \prod S_{nl}^t(\Lambda^{-1})$$

**Follows from field covariance and vacuum invariance**

## Relation to Poincaré invariant quantum mechanics

$$\langle \mathbf{f} | \mathbf{g} \rangle^* = \langle \mathbf{g} | \mathbf{f} \rangle \quad \langle \mathbf{g} | \mathbf{g} \rangle \geq 0$$

$$W^\dagger = W \quad W \geq 0$$



$$W = A^\dagger A \quad |\Psi\rangle = A|\mathbf{g}\rangle$$

$$A = E\sqrt{W} \quad E^\dagger E = I$$

**A: Insert a complete set of Poincaré irreducible eigenstates between the initial and final fields in  $W$**

$$\langle (m, j)\mathbf{p}, \mu; d | A | x_n, \dots x_1 \rangle := \langle (m, j)\mathbf{p}, \mu; d | \phi(x_n) \cdots \phi(x_1) | 0 \rangle$$

$$W_{in} = \int A_i^\dagger | (m, j)\mathbf{p}, \mu; d \rangle d\mathbf{p} dm \langle (m, j)\mathbf{p}, \mu; d | A_n$$

## Examples:

### free scalar field

$$\langle m, \mathbf{p} | A | x \rangle = \frac{e^{-i\omega_m(\mathbf{p})x^0 - \mathbf{p} \cdot \mathbf{x}}}{\sqrt{2\omega_m(\mathbf{p})}}$$

### free spinor field

$$\langle (m, 1/2), \mathbf{p}, \mu | A | x, i \rangle = \frac{e^{-i\omega_m(\mathbf{p})x^0 - \mathbf{p} \cdot \mathbf{x}}}{\sqrt{2\omega_m(\mathbf{p})}} u_{\mu, i}(\mathbf{p})$$

### general case

$$|(m, j), \mathbf{p}, \mu, \mathbf{g}\rangle =$$

$$\int \mathcal{A} U(R, 0) U(I, x) | \mathbf{g} \rangle e^{iR^{-1} \mathbf{p} \cdot \mathbf{x}} D_{\mu\mu}^{j*}(R) \delta(p^2 + m^2) \theta(p^0) d^4 x dR dp^0$$

## Connection with Poincaré invariant quantum mechanics

$$\Psi_{jm}(\mathbf{P}, \mu, \mathbf{g}) = \langle (j, m)_{\mathbf{P}, \mu, d} | A | \mathbf{g} \rangle$$

$$AU_{field}(\Lambda, a) = U_{PI}(\Lambda, a)A \quad \text{PI=Poincaré irreducible}$$

$$\langle (m, j)_{\mathbf{p}, \mu; d} | A | \Lambda x_1 + a, \dots, \Lambda x_n + a \rangle \prod S_i^t(\Lambda^{-1}) =$$

$$\sum_{\mathbf{p}, \mu; \mathbf{p}', \mu'} \mathcal{D}_{\mathbf{p}, \mu; \mathbf{p}', \mu'}^{jm}(\Lambda, a) \langle (m, j)_{\mathbf{p}', \mu'; d} | A | x_n \dots x_1 \rangle$$

$$\mathcal{D}_{\mathbf{p}, \mu; \mathbf{p}', \mu'}^{jm}(\Lambda, a) := \langle (m, j)_{\mathbf{p}, \mu} | U(\Lambda, a) | (m, j)_{\mathbf{p}', \mu'} \rangle$$



$$(U_{PI}(\Lambda, a)\Psi_{jm})(\mathbf{p}, \mu; d) = \sum_{\mathbf{p}', \mu'} \mathcal{D}_{\mathbf{p}, \mu; \mathbf{p}', \mu'}^{jm}(\Lambda, a)\Psi_{jm}(\mathbf{p}', \mu'; d)$$

**Identical to transformation properties of wave functions in Poincaré invariant quantum theory.**

$\infty$  # degeneracy parameters,  $d$ , in field theory case.

$$A = E\sqrt{W} \rightarrow \Psi_{jm}(\mathbf{p}, \mu; d)$$

$\Psi_{jm}(\mathbf{p}, \mu; d)$  determined up to a unitary transformation,  $E$

## Properties of $U_{PI}(\Lambda, a)$

### Cluster properties +

$$W = A^\dagger A$$

$$\lim_{y^2 \rightarrow \infty} W(x_1, \dots, x_n, x_{n+1} + y, \dots, x_m + y) =$$

$$W(x_1 \cdots x_n) W(x_{n+1}, \dots, x_m)$$

$$W = A^\dagger A \rightarrow \prod_{a_i} W_{a_i} = \prod_{a_i} A_{a_i}^\dagger A_{a_i}$$

Same  $E$  in  $A$  and  $A_{a_i}$

$\Downarrow$

$$\lim_{y^2 \rightarrow \infty} \|(U_{PI}(\Lambda, a) - \otimes U_{PI, a_i}(\Lambda, a))AT(y)|\mathbf{g}\| \rightarrow 0$$

## Field theory $\rightarrow$ three-body models

$$\Psi_{jm}(\mathbf{P}, \mu, \mathbf{d})$$

$$\mathbf{d} \in S_{finite} \cup S_{remainder}$$

$$\mathcal{H} = \mathcal{H}_{S_{finite}} \oplus \mathcal{H}_{S_{remainder}}$$

$$U_{PI}(\Lambda, a) : \mathcal{H}_{S_{finite}} \rightarrow \mathcal{H}_{S_{finite}}$$

**Gives exactly Poincaré invariant few degree of freedom system**

## Three-body models

- **Model assumption: Limit degeneracy parameters,  $d$ , corresponding to three-body degrees of freedom**

$$\mathcal{H}_{PI} \rightarrow \mathcal{H}_{3B} \quad U_{PI}(\Lambda, a) \rightarrow U_{3B}(\Lambda, a)$$

- $U_{PI}(\Lambda, a)$  defined up to overall  $E = A(W)^{-1/2}$
- **Construct a model  $U_{3B}(\Lambda, a)$  of  $U_{PI}(\Lambda, a)$  on  $\mathcal{H}_{NB}$  that is consistent with:**
  - three-body experiment
  - $U_{3B}(\Lambda, a)$  unitary rep of Poincaré group.
  - relevant cluster properties.
  - spectral condition.

## Construction of $\mathcal{H}_{NB}$ ( $N = 2, 3$ )

- Identify experimentally observable single-particle DOF ( $m_i, j_i$ )

- Let

$\mathcal{H}_{mj} :=$  mass  $m$  spin  $j$  irreducible representation space

- Construct  $\mathcal{H}_{NB}$   $N = (2, 3)$

$$\mathcal{H}_{3B} := \mathcal{H}_{m_1 j_1} \otimes \mathcal{H}_{m_2 j_2} \otimes \mathcal{H}_{m_3 j_3}$$

$$\mathcal{H}_{2B} := \mathcal{H}_{m_i j_i} \otimes \mathcal{H}_{m_k j_k}$$

Choose a basis for  $\mathcal{H}_{m_j j_j}$  and  $\mathcal{H}_{3B}$

$\{\mathbf{j}^2, m, \mathbf{h}\}$  maximal set of independent commuting Hermitian functions of one-body Poincaré generators

$$\mathbf{h} \rightarrow (\mathbf{p}, \mu), (\mathbf{v}, \mu), (p^+, \mathbf{p}_\perp, \mu_f), \dots$$

$$\mathcal{H}_{m_j j_j} = \{ \Psi_{mj}(\mathbf{h}) \mid \int |\Psi_{mj}(\mathbf{h})|^2 d\mathbf{h} < \infty \}$$

$$(U(\Lambda, a)\Psi_{mj})(\mathbf{h}) = \int \mathcal{D}_{\mathbf{h}, \mathbf{h}'}^{mj}(\Lambda, a) \Psi_{mj}(\mathbf{h}')$$

$$\mathcal{D}_{\mathbf{h}, \mathbf{h}'}^{mj}(\Lambda, a) = \langle (m, j)\mathbf{h} \mid U_{mj}(\Lambda, a) \mid (m, j)\mathbf{h}' \rangle$$

**Construct non-interacting dynamics ( $N = 3$ ),**

$$U_0(\Lambda, a) : \mathcal{H}_{3B} \rightarrow \mathcal{H}_{3B}$$

$$U_0(\Lambda, a) := U_{m_1j_1}(\Lambda, a) \otimes U_{m_2j_2}(\Lambda, a) \otimes U_{m_3j_3}(\Lambda, a)$$

$$(U_0(\Lambda, a)\Psi_{m_1j_1m_2j_2m_3j_3})(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3) =$$

$$\int \prod \mathcal{D}_{\mathbf{h}_i, \mathbf{h}'_i}^{m_i j_i}(\Lambda, a) \Psi_{m_1j_1m_2j_2m_3j_3}(\mathbf{h}'_1, \mathbf{h}'_2, \mathbf{h}'_3) \prod d\mathbf{h}'_i$$

**Construction of dynamical  $U_{2B}(\Lambda, a) : \mathcal{H}_{2B} \rightarrow \mathcal{H}_{2B}$**

**Construct irreducible basis for  $U_0(\Lambda, a)$  in  $(m, j, \mathbf{h})$ -basis using Poincaré group Clebsch-Gordan coefficients**

$$\langle 1, 2 | (12) \rangle = \langle (m_1, j_1), \mathbf{h}_1, (m_2, j_2), \mathbf{h}_2 | (m_{12}, j_{12}), \mathbf{h}_{12}, d_{12} \rangle$$

$$\int \mathcal{D}_{\mathbf{h}_1 \mathbf{h}'_1}^{m_1 j_1}(\Lambda, a) \mathcal{D}_{\mathbf{h}_2 \mathbf{h}'_2}^{m_2 j_2}(\Lambda, a) \langle 1', 2' | (12) \rangle = \int \langle 1, 2 | (12)' \rangle \mathcal{D}_{\mathbf{h}'_{12} \mathbf{h}_{12}}^{m_{12} j_{12}}(\Lambda, a)$$

## Generalized Bakamjian-Thomas construction of

$$U_{bt}(\Lambda, a) : \mathcal{H}_{NB} \rightarrow \mathcal{H}_{NB}$$

Define an interacting mass operator (rest Hamiltonian),  $M$ , by adding an interaction,  $v$ , to the non-interacting mass operator,  $M_0$ , **in the non-interacting irreducible basis.**

Choose  $v$  to satisfy

$$\langle (m, j), \mathbf{h}, \mathbf{d} | v | (m', j'), \mathbf{h}', \mathbf{d}' \rangle = \delta[\mathbf{h} - \mathbf{h}'] \delta_{jj'} \langle m, \mathbf{d} | v^j | m', \mathbf{d}' \rangle$$

$$M_I = M_0 + v$$

Diagonalize  $M$  in this basis to construct simultaneous eigenstates of  $M, j^2, \mathbf{h}$

$$\langle (m, j) \mathbf{h}, \mathbf{d} | (\lambda, j') \mathbf{h}' \rangle = \delta[\mathbf{h} - \mathbf{h}'] \delta_{jj'} \Psi_{\lambda j}(m, \mathbf{d})$$

$$(\lambda - m) \Psi_{\lambda j d \lambda}(m, \mathbf{d}) = \sum_j \langle m, \mathbf{d} | v^j | m', \mathbf{d}' \rangle \Psi_{\lambda j d \lambda}(m', \mathbf{d}')$$

$$M = M^\dagger \quad \Rightarrow \quad \{ |(\lambda, j)\mathbf{h}, d_\lambda\rangle \} \quad \text{complete}$$

$$(U_{bt}(\Lambda, a)\Psi_{\lambda j})(\mathbf{h}, \mathbf{d}_\lambda) := \int \mathcal{D}_{\mathbf{h}\mathbf{h}'}^{\lambda j}(\Lambda, a)\Psi_{\lambda j}(\mathbf{h}', \mathbf{d}_\lambda)$$

$U_{bt}(\Lambda, a)$  unitary

**Defines irreducible dynamical unitary representation,  
 $U_{bt}(\Lambda, a)$ , of the Poincaré group on  $\mathcal{H}_{NB}$**

## Connection with two-body experiment

$$M_0 := \sqrt{\mathbf{k}^2 + m_1^2} + \sqrt{\mathbf{k}^2 + m_2^2} \quad (\text{defines } \mathbf{k}^2)$$

$$M := \sqrt{\mathbf{k}^2 + 2\mu v_{nr} + m_1^2} + \sqrt{\mathbf{k}^2 + 2\mu v_{nr} + m_2^2}$$

$v_{nr}$  **standard non-relativistic interaction**

$$v := M - M_0$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$H_{nr} = \frac{\mathbf{p}^2}{2(m_1 + m_2)} + h_{nr} \quad h_{nr} = \frac{\mathbf{k}^2}{2\mu} + v_{nr}$$

$$M = \sqrt{2\mu h_{nr} + m_1^2} + \sqrt{2\mu h_{nr} + m_2^2}$$

## Two-body scattering

$$\lim_{t \rightarrow \pm\infty} \|e^{-iH_{nr}t}|\psi_{\pm}(0)\rangle - e^{-iH_{nr0}t}|\psi_0(0)\rangle\| = 0$$



$$\lim_{t \rightarrow \pm\infty} \|\psi_{\pm}(0)\rangle - e^{iH_{nr}t}e^{-iH_{nr0}t}|\psi_0(0)\rangle\| = 0$$

$$\lim_{t \rightarrow \pm\infty} e^{iH_{nr}t}e^{-iH_{nr0}t} = \lim_{t \rightarrow \pm\infty} e^{ih_{nr}t}e^{-ih_0t} = \lim_{t \rightarrow \pm\infty} e^{iMt}e^{-iM_0t}$$

$$\lim_{t \rightarrow \pm\infty} e^{iM^2t}e^{-iM_0^2t} = \lim_{t \rightarrow \pm\infty} e^{iH_r^2t}e^{-iH_{r0}^2t} = \lim_{t \rightarrow \pm\infty} e^{iH_r t}e^{-iH_{r0}t}$$

$$S_{fi(nr)}(\mathbf{k}^2) = \langle\psi_+|\psi_-\rangle = S_{fi(r)}(\mathbf{k}^2)$$

## Choice of interaction? - Ekstein's Theorem

$$M' := EME^\dagger \quad M'_0 := M_0 \quad EE^\dagger = I$$

$$S = \Omega_+^\dagger(M, M_0)\Omega_-(M, M_0) \quad S' = \Omega_+^\dagger(M', M_0)\Omega_-(M', M_0)$$

$$S = S' \Leftrightarrow \lim_{t \rightarrow \pm\infty} \|(E - I)e^{-iM_0 t}|\psi\rangle\| = 0$$

$$E = \Omega_\pm(M', M_0)\Omega_\pm^\dagger(M, M_0) + |B'\rangle\langle B|$$

$$M - M_0 = v \sim \bar{v}' = EME^\dagger - M_0$$

$\sim$  = “scattering equivalent”

(Ekstein - 1960)

**Interactions only determined up to unitary transformations,  $E$ , that preserve the  $S$  matrix.**

**Freedom to choose  $E$  appears in  $A = E\sqrt{W}$ .**

**No loss of generality in using**

$$M_{bt} = \sqrt{\mathbf{k}^2 + 2\mu v_{nr} + m_1^2} + \sqrt{\mathbf{k}^2 + 2\mu v_{nr} + m_2^2}$$

**in two-body problem**

**There are NO relativistic corrections for  $N = 2$ .**

## Kinematic subgroups and Dirac forms of dynamics

$(\Lambda, a) \in$  kinematic subgroup associated with irreducible representation basis  $\mathfrak{h}$  if

$$\mathcal{D}_{\mathfrak{h}, \mathfrak{h}'}^{mj}(\Lambda, a)$$

is **independent** of  $m$

Each of Dirac's kinematic subgroups is associated with a different irreducible representation basis,  $(m, j, \mathfrak{h})$ .

Most bases have the identity as the kinematic subgroup

## Kinematic subgroups are scattering equivalent

$$\langle (m, j)\mathbf{h}, \mathbf{d} | (m', j')\mathbf{f}', \mathbf{d}' \rangle =$$

$$\delta(m - m') \delta_{jj'} \delta_{\mathbf{d}\mathbf{d}'} \delta[\mathbf{h} - \mathbf{h}(j, m, \mathbf{f})] \left| \frac{\partial \mathbf{h}}{\partial \mathbf{f}} \right|^{1/2} =$$

$$\delta(m - m') \delta_{jj'} \delta_{\mathbf{d}\mathbf{d}'} \delta[\mathbf{f} - \mathbf{f}(j, m, \mathbf{h})] \left| \frac{\partial \mathbf{f}}{\partial \mathbf{h}} \right|^{1/2}$$

$\left( \left| \frac{\partial \mathbf{f}}{\partial \mathbf{h}} \right|^{1/2} \right)$  is generally mass dependent and has a discrete component

$$\langle (m, j)\mathbf{h}, \mathbf{d} | S | (m', j')\mathbf{h}', \mathbf{d}' \rangle = \delta[\mathbf{h} - \mathbf{h}'] \langle m, \mathbf{d} | S^j | m, \mathbf{d}' \rangle$$

$$\langle (m, j)\mathbf{f}, \mathbf{d} | S | (m', j')\mathbf{f}', \mathbf{d}' \rangle = \delta[\mathbf{f} - \mathbf{f}'] \langle m, \mathbf{d} | S^j | m, \mathbf{d}' \rangle$$

The  $S$ -matrices are identical because the delta functions become equivalent when  $m = m'$

$$N > 2$$

**Cluster properties - separate region  $A$  from region  $B$**



$$\lim_{(x-y)^2 \rightarrow +\infty} \|[U(\Lambda, a) - U_A(\Lambda, a) \otimes U_B(\Lambda, a)] T_A(x) \otimes T_B(x) |\psi\rangle\| \rightarrow 0$$

$$T_C(x) := U_C(I, x) \quad C \in \{A, B\}$$

- **Fails** for  $N = 3$  generalized Bakamjian-Thomas construction!
- **Automatic** for  $2 + 1$  tensor product  $U_{(12)bt}(\Lambda, a) \otimes U_3(\Lambda, a)$

$N = 2 + 1$  - **cluster properties**

$$U_{(12)\otimes(3)}(\Lambda, a) := U_{(12)bt}(\Lambda, a) \otimes U_3(\Lambda, a)$$

$U_{(12)(3)bt}(\Lambda, a)$       **generalized BT**

$$U_{(12)\otimes(3)}(\Lambda, a) \neq U_{(12)(3)bt}(\Lambda, a)$$

**however**

$$S_{(12)\otimes(3)} = S_{(12)(3)bt}$$

$\Downarrow$

$$U_{(12)\otimes(3)}(\Lambda, a) = E_{(12)(3)} U_{(12)(3)bt}(\Lambda, a) E_{(12)(3)}^\dagger$$

## Cluster Properties (2 + 1)

$$\begin{array}{ccc}
 U_{(12)_0}(\Lambda, a) \otimes U_3(\Lambda, a) & \xrightarrow{\langle AB|C \rangle_0} & U_{(12)_0(3)}(\Lambda, a) \\
 \downarrow V_{(12)bt} & & \downarrow V_{(12)(3)bt} \\
 U_{(12)bt}(\Lambda, a) \otimes U_3(\Lambda, a) & \xrightarrow{\langle AB|C \rangle_I} & U_{(12)\otimes(3)}(\Lambda, a) \underbrace{\sim}_{E_{12}} U_{(12)(3)bt}(\Lambda, a)
 \end{array}$$

$$S_{(12)\otimes(3)} = S_{(12)(3)bt} \underbrace{\Rightarrow}_{\text{Ekstein}} U_{(12)(3)bt}(\Lambda, a) = E_{12}^\dagger U_{(12)\otimes(3)}(\Lambda, a) E_{12}$$

## Constructing $\Psi_{jm}(\mathbf{h})$ , $U_{PI}(\Lambda, a)$

$$\mathbf{k}_{ij}^2 : m_{ij} = \sqrt{m_i^2 + \mathbf{k}_{ij}^2} + \sqrt{m_j^2 + \mathbf{k}_{ij}^2}$$

$$\mathbf{q}_k^2 : M_0 = \sqrt{m_{ij}^2 + \mathbf{q}_k^2} + \sqrt{m_k^2 + \mathbf{q}_k^2}$$

$$m_{(ij)bt} = \sqrt{m_i^2 + \mathbf{k}_{ij}^2 + 2\mu_{ij}v_{nr(ij)}} + \sqrt{m_j^2 + \mathbf{k}_{ij}^2 + 2\mu_{ij}v_{nr(ij)}}$$

$$M_{(ij)(k)bt} = \sqrt{m_{(ij)bt}^2 + \mathbf{q}_k^2} + \sqrt{m_k^2 + \mathbf{q}_k^2}$$

## Constructing $\Psi_{jm}(\mathbf{h})$ , $U_{PI}(\Lambda, a)$ : BT interactions

$$V_{(ij)(k)bt} := M_{(ij)(k)bt} - M_0$$

with

$$\langle \mathbf{q}'_k, \mathbf{k}'_{ij}, j'^2, \mathbf{h}', j'_{ij}, \mathbf{d}'_{ij}, \mathbf{d}'_k | v_{nr(ij)} | \mathbf{q}^2_k, \mathbf{k}^2_{ij}, j^2, \mathbf{h}, \mathbf{d}_{ij}, \mathbf{d}_k \rangle =$$

$$\delta[\mathbf{h}' - \mathbf{h}] \delta_{jj'} \delta_{\mathbf{d}'_k \mathbf{d}_k} \delta(\mathbf{q}'_k - \mathbf{q}^2_k) \delta_{jij'} \langle \mathbf{k}'_{ij}, \mathbf{d}'_{ij} | v_{nr}^{jj} | \mathbf{k}^2_{ij}, \mathbf{d}_{ij} \rangle$$

## Constructing $\Psi_{jm}(\mathbf{h}), U_{PI}(\Lambda, a)$

$$M_{bt} = M_{(12)(3)bt} + M_{(23)(1)bt} + M_{(31)(2)bt} - 2M_0 + V_{(123)bt} =$$

$$E_{(12)(3)}^\dagger M_{(12)\otimes(3)} E_{(12)(3)} + E_{(23)(1)}^\dagger M_{(23)\otimes(1)} E_{(23)(1)} +$$

$$E_{(31)(2)}^\dagger M_{(31)\otimes(2)} E_{(31)(2)} - 2M_0 + V_{(123)bt}$$

**Diagonalize in a non-interacting irreducible basis**

⇓

$$U_{bt}(\Lambda, a)$$

$$E(V) := e^{\ln(E_{(12)(3)}) + \ln(E_{(23)(1)}) + \ln(E_{(31)(2)})}$$

$$U_{PI}(\Lambda, a) = E(V)U_{bt}(\Lambda, a)E^\dagger(V) \quad M_{PI} = E(V)M_{bt}E^\dagger(V)$$

$E(V)$  preserves all kinematic symmetries

$U_{PI}(\Lambda, a)$  same  $S$ -matrix as  $U_{bt}(\Lambda, a)$  ( $N = 3$  only).

$$M_{PI} \rightarrow M_{(ij)\otimes(k)} \rightarrow M_0$$

$$U_{PI}(\Lambda, a) \rightarrow U_{ij}(\Lambda, a) \otimes U_k(\Lambda, a) \rightarrow U_i(\Lambda, a) \otimes U_j(\Lambda, a) \otimes U_k(\Lambda, a)$$

## Properties of $U_{PI}(\Lambda, a)$

- Same transformation properties as  $U_{PI}$  derived from field theory.
- Same cluster properties as  $U_{PI}$  derived from field theory.
- Experimental cross sections - agree with correct field theory **when few-body degrees of freedom d dominate** in field theory.
- Same spectral condition as field theory provided  $M_{bt} > 0$
- Both field theory and QM  $U_{PI}(\Lambda, a)$  defined up to scattering equivalence  $E$

**Microcausality ?**

**Relativistic effects?**

## Microcausality (assume $(\mathbf{x} = 0, t = 0)$ Lorentz invariant)

$$U(\Lambda, 0)|\mathbf{0}, 0\rangle_x = |\mathbf{0}, 0\rangle_x \quad U(\Lambda, 0)|\mathbf{p}\rangle_p = |\Lambda\mathbf{p}\rangle_p \frac{\sqrt{\omega(\Lambda\mathbf{p})}}{\sqrt{\omega(\mathbf{p})}}$$

$${}_x\langle\mathbf{0}, 0|\mathbf{p}\rangle_p = {}_x\langle\mathbf{0}, 0|U(\Lambda^{-1}(\mathbf{p}), 0)|\mathbf{p}\rangle_p = {}_x\langle\mathbf{0}, 0|\mathbf{0}\rangle_p \sqrt{\frac{m}{\omega_m(\mathbf{p})}} = \frac{c}{\sqrt{\omega_m(\mathbf{p})}}$$

$$\langle\mathbf{x}|\mathbf{p}\rangle = \langle\mathbf{0}|e^{-i\mathbf{p}\cdot\mathbf{x}}|\mathbf{p}\rangle = \frac{ce^{-i\mathbf{p}\cdot\mathbf{x}}}{\sqrt{\omega_m(\mathbf{p})}}$$

$${}_x\langle\mathbf{x}|\mathbf{0}\rangle_x = \int_x \langle\mathbf{x}|\mathbf{p}\rangle_p \frac{d\mathbf{p}}{2\omega_m(\mathbf{p})}_p \langle\mathbf{p}|\mathbf{0}\rangle_x = c^2 \int d\mathbf{p} \frac{e^{-i\mathbf{p}\cdot\mathbf{x}}}{2\omega_m(\mathbf{p})} = \frac{i}{2} D^+(0, \mathbf{x})$$

$$\frac{i}{2} D^+(0, \mathbf{x}) \rightarrow (m|\mathbf{x}|)^{-1/2} e^{-m|\mathbf{x}|}$$

**No consistent position operators in the enveloping algebra of the Poincaré generators to localize particles in regions smaller than their Compton wavelength ( $\lambda_c = 1/m$ ).**

**$S$  matrix elements at bounded energy **cannot** distinguish microcausal theories from other causal theories.**

**Not enough operators in the model to describe an experiment that can localize particles in space and time;  
error  $\sim \lambda_c$ .**

## Relativistic effects?

- **First appear for  $N=3$**
- **Depend on choice of  $E(V)$**
- **$E(V)$  mixes off-shell effects, three-body force effects with relativistic effects.**
- **Compare to NR three-body model with equivalent two-body interactions.**

## Three-body scattering

$$S_{ac} = S_{ac,bt} = \delta_{ac} - 2\pi i \delta(M_a - M_c) T_{bt}^{ac}$$

$$T_{bt}^{ac}(z) = T_{bt}^{ac}(z) = V_{bt}^c + V_{bt}^a R_{bt}(z) V_{bt}^c$$

$$R_{bt}(z) = (z - M_{bt})^{-1} \quad V_{a,bt} = M_{a,bt} - M_0 \quad V_{bt}^c = \sum_{a \neq b} V_{a,bt}$$

$$a, b, c \in \{(12)(3), (23)(1), (31)(2)\}$$

## Faddeev equation

$$R_{c,bt}(z) = (z - M_0 - V_{c,bt})^{-1}$$

$$R_{bt}(z) = R_{c,bt}(z) + R_{c,bt}(z)V_{bt}^c R_{bt}(z)$$

$$T_{bt}^{ab}(z) = V_{bt}^b + \sum_{c \neq a} V_{c,bt} R_{c,bt}(z) T_{bt}^{cb}(z)$$

## Relativistic effects

$$\langle a | T_{bt}^{ab}(z) | b \rangle = \langle a | V_{bt}^b | b \rangle + \sum_{c \neq a} \langle a | c \rangle \langle c | \bar{V}_{c,bt} \bar{R}_{c,bt}(z) | c \rangle \langle c | \bar{T}_{bt}^{cb}(z) | b \rangle$$

$\langle a | c \rangle =$  **Poincaré Group Racah coefficient**

$$\langle a | c \rangle = \delta[\mathbf{h} - \mathbf{h}'] \delta(M - M') \delta_{JJ'} \mathcal{R}^{MJ}(d_a, d_c)$$

**Replaces NR permutation operator**

## Relativistic effects

$$z = z_c$$

$$\langle c' | V_{c,bt} R_{c,bt}(z_c) | c \rangle = \langle c' | T_{c,bt}(z_c) (z_c - M_0)^{-1} | c \rangle =$$

$$\langle c' | V_{c,bt} | c^- \rangle (z_c - M_0)^{-1} = \langle c' | M_{c,bt} - M_0 | c^- \rangle (z_c - M_0)^{-1} =$$

$$\frac{2\mu}{\omega_1 \omega_2 + \omega'_1 \omega'_2} \frac{(\omega_1 + \omega_2)^2 + (\omega'_1 + \omega'_2)^2}{\omega_1 + \omega_2 + \omega'_1 + \omega'_2} \langle c | t_{nr}(k) | c' \rangle (z_c - M_0)^{-1}$$

$$\omega_i = \sqrt{\mathbf{k}^2 + m_i^2}$$

**Large mass limit**  $\rightarrow t_{nr}/(z_{c,nr} - h_{nr})$

## Relativistic effects

$$z \neq z_c$$

### First resolvent equation

$$T_c(z') = T_c(z_c) + T_c(z') \frac{(z' - z_c)}{(z' - M_0)(z_c - M_0)} T_c(z_c)$$

$$\langle c' | T_c(z_c) | c \rangle = \frac{4m}{m_c + m'_c} \langle c' | t_{NN}(z_c) | c \rangle$$

**Input to Faddeev kernel**

## Relativistic effects

- **Permutation operators.**
- **Modification of half-shell  $t$ .**
- **Modification of free resolvent.**

## Predictions

### Binding energies

$$M|\Psi\rangle = \lambda|\Psi\rangle$$

$$|\Psi\rangle = E(V)|\Psi_{bt}\rangle \quad M_{bt}|\Psi_{bt}\rangle = \lambda|\Psi_{bt}\rangle$$

### Scattering probabilities ( $N = 3$ only)

$$|S_{fi}|^2 = |\langle\Psi_f^+|\Psi_i^-\rangle|^2 = |\langle\Psi_{fbt}^+|\Psi_{bti}^-\rangle|^2$$

### Electromagnetic and weak current matrix elements

$$\langle\Psi_f|I^\nu(0)|\Psi_i\rangle = \langle\bar{\Psi}_f|E^\dagger(V)I^\nu(0)E(V)|\bar{\Psi}_i\rangle$$

## Low-energy results (with spin)

### Triton binding energy

H.Kamada et. al. EFB 20 Proceedings

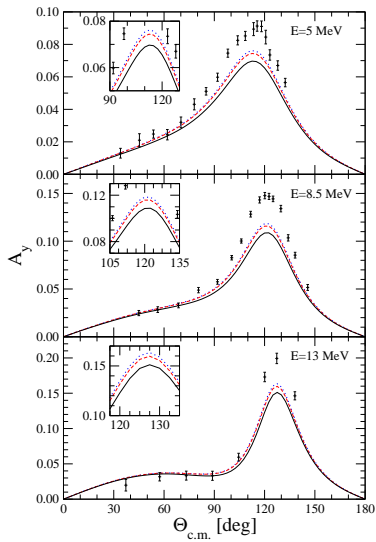
$$A_y$$

H. Witała et. al., Phys. Rev. C77, 034004 (2008)

G. A. Miller and A. Schwenk, Phys. Rev. C76, 024001  
(2007)

## Triton Binding Energy

Potential	$E_b^{nr}$	$E_b^r$	$\Delta$
RSC	-7.02	-6.97	0.05
CD-Bonn	-8.33	-8.22	0.11
Nijmegen II	-7.65	-7.58	0.07
Nijmegen I	-8.00	-7.90	0.10
Nijmegen 93	-7.76	-7.68	0.08
AV18	-7.66	-7.59	0.07
Exp. (-8.48)			



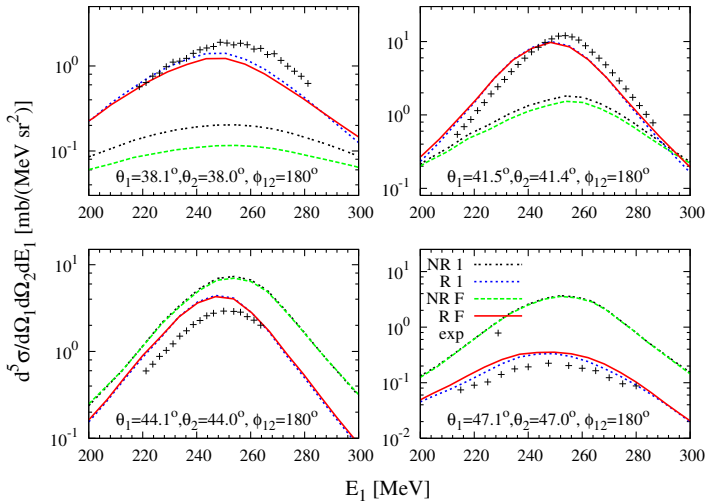
**Results up to 2 GeV (no spin)**

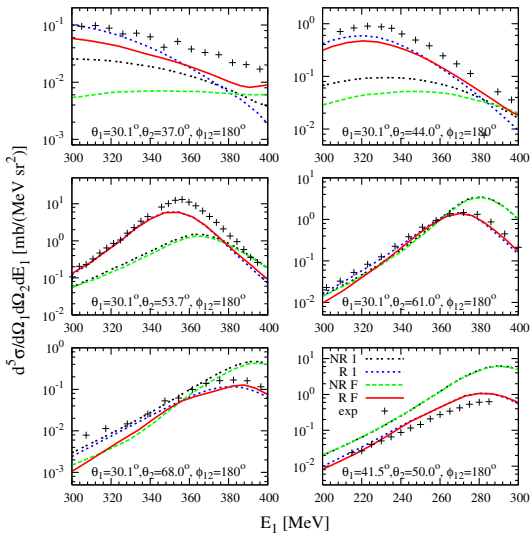
**Exclusive breakup at 500 MeV**

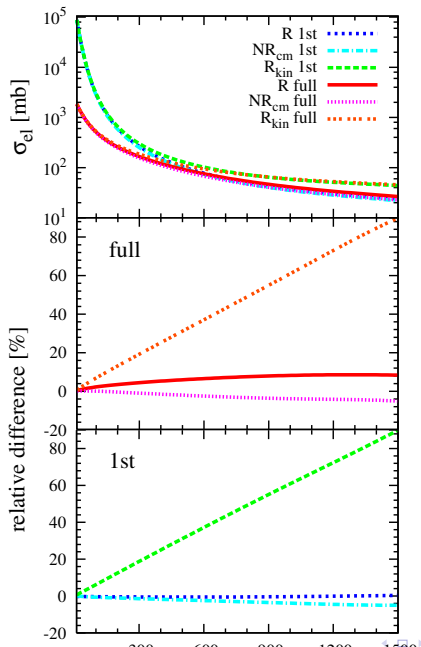
**T. Lin et. al. Physics Letters B660(2008),345,**

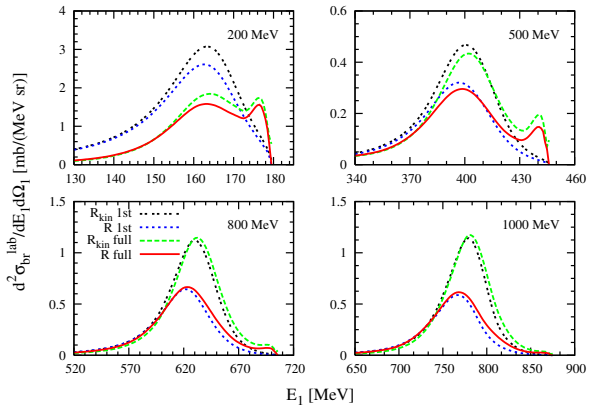
**Tests of approximations**

**T. Lin et. al., nucl-th/0801.3210,**









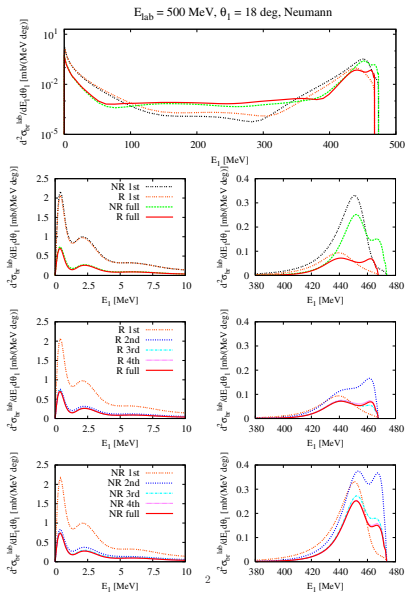
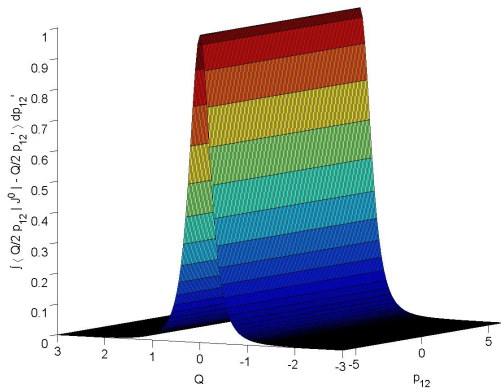


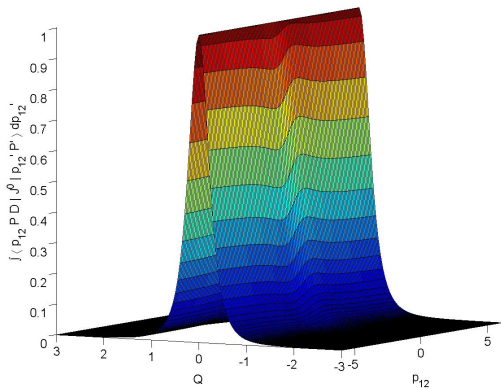
Figure 2: The inelastic differential cross section at projectile energy  $E_{\text{lab}} = 500 \text{ MeV}$  and  $\theta_1 = 18^\circ$ . Neumann

**Need for cluster properties for  $N \geq 4$  (electron scattering  
-Keister, Tucker)**

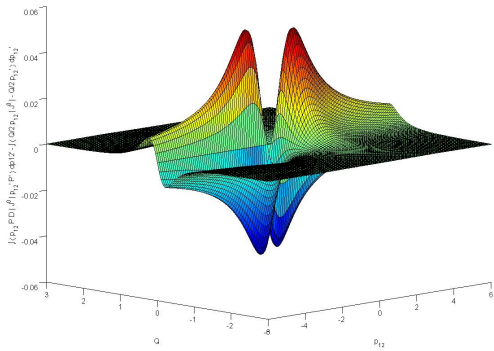
Parallel TP



Parallel BT



Paralel BT - TP



## Conclusion - Outlook

- **Restriction of degrees of freedom in QFT constrains structure of relativistic quantum mechanical models.**
- **Freedom to change representation allows one to utilize existing realistic NR interactions.**
- **Connection with QFT determines the structure of extensions to more complex reactions.**
- **Cluster properties has observable consequences for systems of four or more particles.**
- **Realistic few-body calculations are possible, relativistic effects first appear for  $N = 3$ .**