Wavelets in field theory

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Abstract

We discuss the use of Daubechies wavelets in discretizing quantum field theories.

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1 Introduction

Daubechies wavelets [1][2] and the associated scaling functions are a basis for the square integrable functions on the real line that have properties that make them useful for discretizing quantum field theories [3]. These properties include:

1. The basis functions have compact support.
2. Subsets of the basis functions form locally finite partitions of unity.
3. The basis functions are related to fixed points of a renormalization group equation.
4. The basis functions are natural for treating problems with multiple scales.
5. The basis functions have a limited amount of smoothness.

Wavelets methods have been discussed for applications to quantum field theory both from a computational [4] [5][6][7][8] and theoretical [9][10][11][12][13] perspective. In this work we provide a general discussion of the properties of fields smeared with compactly supported wavelets. The multiscale structure of the wavelet basis leads to a natural framework for eliminating short-distance degrees of freedom, resulting in effective models that are amenable to computation.

2 Scaling functions and renormalization group equations

The basis functions are constructed from a single function using a unitary dyadic scale transformation D and a unit translation operator T defined by:

$$\underbrace{(Df)(x) = \sqrt{2}f(2x)}_{\text{scale change}} \qquad \underbrace{(Tf)(x) = f(x-1)}_{\text{translation}}. \tag{1}$$

Application of the operator D to a function reduces the volume of its support by a factor of two and adjusts the scale to preserve the $L^2(\mathbb{R})$ norm. The operator T is a discrete translation.

All of the basis functions are finite linear combinations of translated and rescaled copies of a single function, called the scaling function, $s(x)$. The scaling function is a solution of the scaling equation

$$s(x) = D(\sum_{l=0}^{2K-1} h_l T^l s(x)). \tag{2}$$

Since (2) is a homogeneous equation, the solution is fixed by the scale fixing condition

$$\int s(x)dx = 1. \tag{3}$$

The renormalization group structure of the scaling equation is illustrated by

$$f_n(x) = \underbrace{D\,\underbrace{(\sum_{l=0}^{2K-1} h_l T^l f_{n-1}(x))}_{\textbf{block average}}}_{\textbf{rescale}}\,. \tag{4}$$

Comparing (2) to (4) shows that $s(x)$ is a fixed point of this renormalization group equation. The coefficients h_l in (2) and (4) are fixed constants that determine properties of the fixed point, $s(x)$. They are solutions of the following set of algebraic equations:

$$\sum_{n=0}^{2K-1} h_n = \sqrt{2} \qquad \sum_{n=0}^{2K-1} h_n h_{n-2m} = \delta_{m0} \qquad \sum_{n=0}^{2K-1} n^m (-l)^n h_{2K-1-n} = 0 \qquad m < K \tag{5}$$

where K is an integer that defines the type of Daubechies scaling function. The first equation is a necessary condition for the scaling equation to have a solution, the second equations ensure that unit translates of the scaling function are orthonormal, and the third equations ensure that x^m for $m < K$ can be pointwise represented by a locally finite linear combination of scaling functions. Exact solutions of equations (**??**) for $K = 1, 2$ and 3 can be found in [3]. The choice of K controls the support, smoothness, and the number of non-zero translated scaling functions at any point.

Scaling functions with different resolutions are obtained by applying integer translations and dyadic rescaling to the original scaling function

$$s^k_n(x) := (D^k T^n s)(x) = \sqrt{2^k} s(2^k(x - n/2^k)). \tag{6}$$

It can be shown that the scaling functions have the following properties:

1. Reality: $s^k_n(x) = s^k_n(x)^*$
2. Compact support: $\text{support}[s^k_n(x)] = 1/2^k[n, 2K - 1 + n]$
3. Orthonormality: $\int s^k_m(x) s^k_n(x)dx = \delta_{mn}$
4. Pointwise low-degree polynomial representation: $x^m = \sum_n c_n(m) s^k_n(x)$ $m < K$, any k
5. Partition of unity: $1 = \sum_{n=-\infty}^{\infty} \sqrt{\frac{1}{2^k}} s^k_n(x - n)$ any k
6. Differentiability (K>2): $\frac{ds^k_n(x)}{dx}$ exists for $K \geq 3$.

The scale-$1/2^k$ scaling functions, $\{s^k_n(x)\}_{n=-\infty}^{\infty}$, are an orthonormal basis for the resolution $1/2^k$ subspace, $\mathcal{S}_k$, of $L^2(\mathbb{R})$ defined by

$$\mathcal{S}_k := \{f(x)|f(x) = \sum_{n=-\infty}^{\infty} c_n s^k_n(x), \quad \sum_{n=-\infty}^{\infty} |c_n|^2 < \infty\}. \tag{7}$$

3 Wavelets and the multiresolution decomposition of the Hilbert space

The scaling equation implies that the $s^k_n(x)$ are linear combinations of $s^{k+1}_n(x)$ leading to the nested inclusions

$$L^2(\mathbb{R}) \supset \cdots \supset \mathcal{S}_{k+1} \supset \mathcal{S}_k \supset \mathcal{S}_{k-1} \supset \cdots . \tag{8}$$

The wavelet subspace, $\mathcal{W}_n$, is defined as the orthogonal complement of $\mathcal{S}_n$ in $\mathcal{S}_{n+1}$:

$$\mathcal{S}_{n+1} = \mathcal{S}_n \oplus \mathcal{W}_n. \tag{9}$$

Repeated application of (9) gives a decomposition of $\mathcal{S}_n$ into orthogonal subspaces with resolutions $1/2^{n-m} \cdots 1/2^n$:

$$\mathcal{S}_n = \mathcal{W}_{n-1} \oplus \mathcal{W}_{n-2} \oplus \cdots \oplus \mathcal{W}_{n-m} \oplus \mathcal{S}_{n-m}. \tag{10}$$

This can be extended to all scales using

$$L^2(\mathbb{R}) = \lim_{n\to\infty} \mathcal{S}_n = \mathcal{S}_k \oplus \mathcal{W}_k \oplus \mathcal{W}_{k+1} \oplus \cdots \tag{11}$$

which gives a decomposition of $L^2(\mathbb{R})$ into a direct sum of orthogonal subspaces of different resolutions. Orthonrmal bases for the wavelet spaces are constructed from the *mother wavelet*, defined by

$$w(x) = D \sum_{l=0}^{2K-1} g_l T^l s(x) \qquad \text{where} \qquad g_l := (-)^l h_{2K-1-l}. \tag{12}$$

The scale k wavelets, $w^k_n(x) := D^k T^n w(x)$, are an orthonormal basis for the subspace $\mathcal{W}_k$:

$$\mathcal{W}_k := \{f(x)|f(x) = \sum_{n=-\infty}^{\infty} c_n w^k_n(x), \quad \sum_{n=-\infty}^{\infty} |c_n|^2 < \infty\}. \tag{13}$$

While the scaling functions can be used to point-wise represent low-degree polynomials, the wavelets are orthogonal to the same low-degree polynomials.

$$\int w^k_n(x) x^m dx = 0 \qquad 0 < m < K. \tag{14}$$

Equation (11) means that the scale $1/2^k$ scaling functions and the wavelets on all finer scales,

$$\{s^k_n(x)\}_{n=-\infty}^{\infty} \cup \{w^{k+l}_n(x)\}_{n=-\infty}^{\infty}{}_{l=0}^{\infty} \tag{15}$$

are an orthonormal basis for the space of square integrable functions on the line. This basis has all of the properties listed in the beginning of these proceedings.

The relation (9) means that $\{s^k_n(x)\}_{n=-\infty}^{\infty}$ and $\{s^{k-1}_n(x)\}_{n=-\infty}^{\infty} \cup \{w^{k-1}_n(x)\}_{n=-\infty}^{\infty}$ are two different orthonormal bases on the same space. They are necessarily related by a real orthogonal transformation called the wavelet transform that is given explicitly by:

$$s^{k-1}_n(x) = \sum_{l=0}^{2K-1} h_l s^k_{2n+l}(x) \qquad w^{k-1}_n(x) = \sum_{l=0}^{2K-1} g_l s^k_{2n+l}(x) \tag{16}$$

$$s^k_n(x) = \sum_m h_{n-2m} s^{k-1}_m(x) + \sum_m g_{n-2m} w^{k-1}_m(x). \tag{17}$$

Because the transformation is orthogonal, the same coefficients appear in the transformation and its inverse.

These relations express a fine-scale basis in terms of a coarse-scale basis, which is related to the fine-scale basis by a unitary scale transformation, and additional functions (wavelets) that fill in the missing fine-scale information. These concepts can be used decompose local fields into linear combinations of discrete fields of different resolutions.

For the purpose of illustration we consider fields in $1+1$ dimensions. Fields on multidimensional spaces can be treated using products of one-dimensional scaling functions and wavelets. We expand a set of local fields $\mathbf{\Phi}(x,t)$, and $\mathbf{\Pi}(x,t)$ satisfying canonical equal time commutation relations:

$$[\mathbf{\Phi}(x,t),\mathbf{\Pi}(y,t)] = i\delta(x-y) \tag{18}$$

in the basis (15).

We define discrete fields by smearing these fields with the basis functions (15)

$$\mathbf{\Phi}^k_s(n,t) := \int s^k_n(x)\mathbf{\Phi}(x,t)dx \qquad \mathbf{\Phi}^l_w(n,t) := \int w^l_n(x)\mathbf{\Phi}(x,t)dx \tag{19}$$

$$\mathbf{\Pi}^k_s(n,t) := \int s^k_n(x)\mathbf{\Pi}(x,t)dx \qquad \mathbf{\Pi}^l_w(n,t) := \int w^l_n(x)\mathbf{\Pi}(x,t)dx. \tag{20}$$

The discrete fields are associated with degrees of freedom of the theory corresponding to different compact regions.

4 Multiresolution fields

The orthonormality of the basis (15) means that the discrete fields also satisfy the equal-time commutation relations

$$[\mathbf{\Phi}^k_s(m,t),\mathbf{\Phi}^k_s(n,t)] = 0 \qquad [\mathbf{\Pi}^k_s(m,t),\mathbf{\Pi}^k_s(n,t)] = 0 \qquad [\mathbf{\Phi}^k_s(m,t),\mathbf{\Pi}^k_s(n,t)] = i\delta_{mn} \tag{21}$$

$$[\mathbf{\Phi}^k_w(m,t),\mathbf{\Phi}^l_w(n,t)] = 0 \qquad [\mathbf{\Pi}^k_w(m,t),\mathbf{\Pi}^l_w(n,t)] = 0 \qquad [\mathbf{\Phi}^k_w(m,t),\mathbf{\Pi}^l_w(n,t)] = i\delta_{mn}\delta_{kl} \tag{22}$$

$$[\mathbf{\Phi}^k_w(m,t),\mathbf{\Phi}^k_s(n,t)] = 0 \qquad [\mathbf{\Pi}^k_w(m,t),\mathbf{\Pi}^k_s(n,t)] = 0 \qquad [\mathbf{\Phi}^k_w(m,t),\mathbf{\Pi}^k_s(n,t)] = 0. \tag{23}$$

These discrete fields form a local algebra in the sense that there are fields with support in arbitrarily small spatial volumes.

The exact fields can be expanded in terms of these discrete operators

$$\mathbf{\Phi}(x,t) = \sum_{n=-\infty}^{\infty} \mathbf{\Phi}^k_s(n,t)s^k_n(x) + \sum_{n=-\infty}^{\infty}\sum_{l=k}^{\infty} \mathbf{\Phi}^l_w(n,t)w^l_n(x), \tag{24}$$

with a similar expression for $\mathbf{\Pi}(x,t)$. These expansions decompose the fields into well-defined operators associated with different resolutions.

We can construct resolution m truncations of these fields by discarding degrees of freedom associated with scales smaller than $1/2^m$:

$$\mathbf{\Phi}^m(x,t) = \sum_{n=-\infty}^{\infty} \mathbf{\Phi}^m_s(n,t)s^m_n(x) = \sum_{n=-\infty}^{\infty} \mathbf{\Phi}^k_s(n,t)s^k_n(x) + \sum_{n=-\infty}^{\infty}\sum_{l=k}^{m-1} \mathbf{\Phi}^l_w(n,t)w^l_n(x), \tag{25}$$

again with an analogous expansion for $\mathbf{\Pi}^m(x,t)$. Truncations on n give volume cutoffs. The advantage of having a basis is that the eliminated degrees of freedom can be systematically restored.

It is possible to replace the discrete Hermitian canonical fields by discrete creation and annihilation operators

$$\mathbf{a}^k_s(n,t) := \frac{1}{\sqrt{2}}(\sqrt{\gamma}\mathbf{\Phi}^k_s(n,t) + i\frac{1}{\sqrt{\gamma}}\mathbf{\Pi}^k_s(n,t)), \tag{26}$$

$$\mathbf{a}^k_w(n,t) := \frac{1}{\sqrt{2}}((\sqrt{\gamma}\mathbf{\Phi}^k_w(n,t) + i\frac{1}{\sqrt{\gamma}}\mathbf{\Pi}^k_w(n,t)) \tag{27}$$

where γ is chosen so $\mathbf{a}^k_s(n,t)$ and $\mathbf{a}^k_w(n,t)$ annihilate the vacuum.

One of the difficulties with field theories is that products of local fields at the same point are ill-defined. On the other hand products of smeared fields are well defined, but because of the smearing they lose their local character. Replacing the fields in the Hamiltonain or Poincaré generators by expansions of the form (24) means that the products of the individual operators are well defined; but because we are writing the exact ill-behaved Hamiltonian as a sum of well-defined operators means that these sums will not converge. On the other hand, if the fields are replaced by finite volume - finite resolution truncations of the type (25), then the truncated theory is well defined.

It is instructive to exhibit the structure of the resolution $1/2^k$ Hamiltonian for a 1+1 dimensional $:\phi^4(x):$ interaction. It has the form

$$H^k = \frac{1}{2}\sum : \big(\mathbf{\Pi}^k_s(n,0)^2 + D^k_{mn}\mathbf{\Phi}^k_s(n,0)\mathbf{\Phi}^k_s(m,0) + \mu^2\mathbf{\Phi}^k_s(n,0)^2 +$$

$$\lambda\Gamma^k_{n_1n_2n_3n_4}\mathbf{\Phi}^k_s(n_1,0)\mathbf{\Phi}^k_s(n_2,0)\mathbf{\Phi}^k_s(n_3,0)\mathbf{\Phi}^k_s(n_4,0)\big) : \tag{28}$$

where the numerical coefficients D^k_{mn} and $\Gamma^k_{n_1n_2n_3n_4}$ are overlap integrals of derivatives of scaling functions or products of scaling functions:

$$D^k_{mn} = \int dx \frac{\partial}{\partial x}s^k_m(x)\frac{\partial}{\partial x}s^k_n(x), \qquad \Gamma^k_{n_1n_2n_3n_4} := \int s^k_{n_1}(x)s^k_{n_2}(x)s^k_{n_3}(x)s^k_{n_4}(x)dx. \tag{29}$$

The advantage of using basis functions that are related to fixed points of a renormalization group equation is that the Hamiltonians with different resolutions have the same form. The only difference is that the numerical coefficients (29) for different scale truncated Hamiltonians differ by powers of 2. For the coefficients (29) these scaling identities are

$$D^k_{mn} = 2^k D^0_{mn}, \qquad \Gamma^k_{n_1\cdots n_m} = 2^{k(m-2)/2}\Gamma^0_{n_1\cdots n_m} \tag{30}$$

so it is only necessary to know these quantities on one scale.

The renormalization group equations imply that scale 1 quantities satisfy finite systems of algebraic equations

$$\Gamma^0_{0n_2n_3} = \sqrt{2}\sum h_{l_1}h_{l_2}h_{l_3}\Gamma^0_{02n_2+l_2-l_1,2n_3+l_3-l_1} \qquad \sum_{n_3}\Gamma^0_{0n_2n_3} = \delta_{n_20} \tag{31}$$

$$D^0_{0,n_1} = \sum 4h_{l_1}h_{l_2}D^0_{0,l_2n_2+l_2-l_1} \tag{32}$$

which can be solved for all of the coefficients that appear in the truncated Hamiltonians at any scale[3].

An important identity is the relation between the discrete field operators on adjacent scales, given by

$$\mathbf{\Phi}^{k+1}_s(n,0) = \sum_m (h_{n-2m}\mathbf{\Phi}^k_s(n,0) + g_{n-2m}\mathbf{\Phi}^k_w(n,0)). \tag{33}$$

Using (27) and (33) in the scale $k+1$ Hamiltonian (28) gives

$$H^{k+1}_s(\mathbf{a}^{k+1}_s, \mathbf{a}^{k+1\dagger}_s) = H^k_s(\mathbf{a}^k_s, \mathbf{a}^{k\dagger}_s) + H^k_w(\mathbf{a}^k_w, \mathbf{a}^{k\,\dagger}_w) + H^k_{sw}(\mathbf{a}^k_s, \mathbf{a}^{k\dagger}_s, \mathbf{a}^k_w, \mathbf{a}^{k\,\dagger}_w) \tag{34}$$

where $H^{k+1}_s(\mathbf{a}^{k+1}_s, \mathbf{a}^{k+1\dagger}_s)$ and $H^k_s(\mathbf{a}^k_s, \mathbf{a}^{k\dagger}_s)$ both have the form (28). $H^k_w(\mathbf{a}^k_w, \mathbf{a}^{k\,\dagger}_w)$ represents fine-scale degrees of freedom that are not coupled to the coarse-scale Hamiltonian, while $H^k_{sw}(\mathbf{a}^k_s, \mathbf{a}^{k\dagger}_s, \mathbf{a}^k_w, \mathbf{a}^{k\,\dagger}_w)$ includes the operators that couple the two scales.

In these expression the coarse and fine-scale Hamiltonians have the same form. The coarse scale Hamiltonain is fixed by adjusting the bare masses and coupling parameters to agree with some coarse scale observables. If one is only interested in coarse-scale observables it is possible to construct a unitary operator that decouples the coarse and fine scale degrees of freedom. This gives a coarse-scale Hamiltonain that involves only explicit coarse-scale degrees of freedom, but has coefficients that include the effects of the eliminated fine scale degrees of freedom. The new coarse-scale Hamiltonian involves the same parameters (bare masses and coupling constants) as the original coarse-scale Hamiltonain. These have to be re-adjusted in order to keep the value of the coarse scale observables unchanged. This can in principle be repeated; at each stage the bare observables need to be adjusted as the effects of additional fine-scale degrees of freedom are included. By absorbing some of the scaling behavior in the mass and coupling constants, one gets renormalized parameters. One gets non-trivial theories if this process leads to finite limits of the renormalized quantities in the limit that the effects of arbitrarily fine scale observables are included.

The elimination of the small-scale degrees of freedom can be attempted using a number of methods, such similarity renormalization group methods

$$\frac{dH(\lambda)}{d\lambda} = [H(\lambda), [H(\lambda), G]] \tag{35}$$

where G is a generator that is chosen to evolve the Hamiltonian to a form that decouples the low and high-resolution degrees of freedom. An advantage in any of these methods is that all of the operators have the structure of known constants with known scaling properties and discrete creation and annihilation operators, so everything is easily computable.

Another important feature of the wavelet truncation of fields takes advantage of the partition of unity property. Noether's theorem gives formal expression for the Poincaré generators in terms of local densities at a fixed time, satisfying

$$[O^a(x), O^b(y)] = i\delta(x-y) f^{abc} O^c(y) \tag{36}$$

where f^{abc} are the structure constants of the Poincaré Lie algebra. Using the partition of unity in the forms

$$1 = (2^{-k/2}\sum_n s_n^k(x))(2^{-k/2}\sum_m s_m^k(y)) \qquad 1 = (2^{-k/2}\sum_n s_n^k(x)) \tag{37}$$

on the left and right respectively, gives the exact identity

$$[\sum_n O_n^{ak}, \sum_m O_m^{bk}] = i f^{abc} \sum_l O_l^{ck} \tag{38}$$

where

$$O_n^{ak} := 2^{k/2} \int O^a(x) s_n^k(x) dx. \tag{39}$$

On the other hand, since $O^a(x)$ involves product of fields, if each field in the product is replaced by the finite resolution approximations,

$$\mathbf{\Phi}(\mathbf{x},t) \to \mathbf{\Phi}^k(\mathbf{x},t) \qquad \mathbf{\Pi}(\mathbf{x},t) \to \mathbf{\Pi}^k(\mathbf{x},t), \tag{40}$$

then O_n^{ak} will be replaced by an approximation that will not satisfy equation (38). Obviously the correct commutation relations must be recovered in the infinite resolution limit. These violations of the symmetry can be studied by looking at the scale of the terms that violate the commutation relations.

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