

# Euclidean Relativistic Quantum Mechanics

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**Collaborators**

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## Motivation

- **Construct relativistic quantum models of systems with a finite number of degrees of freedom**
- **Desirable features:**
  - **Models motivated by field theory**
  - **Cluster properties**

## Elements of relativistic quantum theory

- Hilbert space  $\mathcal{H}$  (requirement for a quantum theory)
- Unitary representation of the Poincaré group (requirement for relativistic invariance of quantum probabilities).
- Cluster properties (required for tests of relativity on isolated subsystems)
- Spectral condition (required for stability of theory)

## **Input**

**Reflection-positive Euclidean Green function(s) or generating functional**

## **Problem**

**Construct relativistic quantum mechanical models**

**(We want to avoid using analytic continuation!)**

## Field theory motivation

**Euclidean generating functional or Green functions:**

$$Z[f] := \frac{\int D_e[\phi] e^{-A[\phi] + i\phi(f)}}{\int D_e[\phi] e^{-A[\phi]}} = \sum_n \frac{(i)^n}{n!} S_n(\underbrace{f, \dots, f}_{n \text{ times}})$$

$A[\phi] = \mathbf{Action}$ ,  $D_e[\phi] = \mathbf{Euclidean "path measure"}$

$f(\tau, \mathbf{x}) = \mathbf{Positive Euclidean-time support test functions}$

$$\mathcal{S}_+ := \{f(\tau, \mathbf{x}) \in \mathcal{S} \mid f(\tau, \mathbf{x}) = 0, \quad \tau < 0\}$$

**Euclidean time reflection**

$$\theta f(\tau, \mathbf{x}) := f(-\tau, \mathbf{x})$$

## Reconstruction of Quantum Mechanics

Osterwalder and Schrader - C.M.P. 31(1973)83;42(1975)281

Fröhlich - Helv. Phys. Acta. 47(1974)265

### Vectors (dense set)

$$B[\phi] = \sum_{j=1}^{N_b} b_j e^{i\phi(f_j)} \quad C[\phi] = \sum_{k=1}^{N_c} c_k e^{i\phi(g_k)}$$

$$b_j, c_k \in \mathbb{C} \quad f_j, g_k \in \mathcal{S}_+ \quad N_b, N_c < \infty$$

### Hilbert space inner product

$$\langle B|C \rangle = \sum_{j=1}^{N_b} \sum_{k=1}^{N_c} b_j^* c_k Z[g_k - \theta f_j]$$

## Remarks

$B[\phi]$  “wave functionals”

$$\langle B|C\rangle = \sum_{j=1}^{N_b} \sum_{k=1}^{N_c} b_j^* c_k Z[g_k - \theta f_j]$$

- The inner product is the **physical (Minkowski)** inner product!
- The generating functional and test functions are **Euclidean!**
- All integrals are over Euclidean space-time variables!
- No analytic continuation is used to calculate the Minkowski scalar product!

**Reflection positivity**  
**(Osterwalder-Schrader Positivity)**

$$\langle B|B \rangle \geq 0$$

**Property of  $Z[f]$  or  $\{S_n(x_1, \dots, x_n)\}$**

$$M_{ij} = Z[f_i - \theta f_j] \geq 0 \quad \forall \quad \{f_1, \dots, f_N\} \in \mathcal{S}_+$$

## Operator algebra

(for scattering asymptotic condition)

$$BC[\phi] = B[\phi]C[\phi] = \sum_{j=1}^{N_b} \sum_{k=1}^{N_c} b_j c_k e^{i\phi(g_k + f_j)} = \sum_{n=1}^{N_c} d_n e^{i\phi(h_n)}$$

$$N_d = N_b N_c \quad h_n = g_k + f_j \quad d_n = b_j c_k$$

## Cluster properties

$$g_{\mathbf{a}}(\tau, \mathbf{x}) := g(\tau, \mathbf{x} - \mathbf{a})$$

$$\lim_{|\mathbf{a}| \rightarrow \infty} (Z[f + g_{\mathbf{a}}] - Z[f]Z[g]) \rightarrow 0$$

$$\lim_{|\mathbf{a}| \rightarrow \infty} S_{m+n}(f, \dots, f, g_{\mathbf{a}}, \dots, g_{\mathbf{a}}) =$$

$$S_m(f, \dots, f)S_n(g, \dots, g)$$

## Operators

$$[T(\beta, \mathbf{a}), B][\phi] := \sum_{j=1}^{N_b} b_j e^{i\phi(f_{j,\beta,\mathbf{a}})}$$

$$f_{n,\beta,\mathbf{a}}(\tau, \mathbf{x}) := f_n(\tau - \beta, \mathbf{x} - \mathbf{a}) \quad \beta > 0$$

$$[U(R), B][\phi] := \sum_{j=1}^{N_b} b_j e^{i\phi(f_{j,R})}$$

$$f_{j,R}(\tau, \mathbf{x}) := f_j(\tau, R\mathbf{x}) \quad f_j \in \mathcal{S}_+ > 0$$

$$[W(\hat{\mathbf{n}}, \psi), B][\phi] := \sum_j c_j e^{i\phi(f_{k,\psi,\hat{\mathbf{n}}})}$$

$$f_{j,\phi,\hat{\mathbf{n}}}(\tau, \mathbf{x}) := f_j(\tau', \mathbf{x}') \quad f_f \in \mathcal{S}_{\chi,+}$$

$$\mathcal{S}_{\chi,+} = \{f \in \mathcal{S}_+ \mid f(\tau, \mathbf{x}) = 0 \quad \tan^{-1}\left(\frac{\tau}{|\mathbf{x}|}\right) \geq \chi\}$$

$$\psi < \pi/2 - \chi$$

$$\tau' = \tau \cos(\psi) - x_{\hat{\mathbf{n}}} \sin(\psi) \quad x'_{\hat{\mathbf{n}}} = x_{\hat{\mathbf{n}}} \cos(\psi) + \tau \sin(\psi)$$

## Poincaré generators

$$[H, B][\phi] = -\frac{\partial}{\partial\beta} (T(\beta, \mathbf{0})B)[\phi]_{\beta=0}$$

$$[\mathbf{P}, B][\phi] = -i\frac{\partial}{\partial\mathbf{a}} (T(0, \mathbf{a})B)[\phi]_{\mathbf{a}=0}$$

$$[(\mathbf{K} \cdot \hat{\mathbf{n}}), B][\phi] = -\frac{\partial}{\partial\psi} (W(\hat{\mathbf{n}}, \psi)B)[\phi]_{\psi=0}$$

$$[(\mathbf{J} \cdot \hat{\mathbf{n}}), B][\phi] = -i\frac{\partial}{\partial\psi} (R(\hat{\mathbf{n}}, \psi)B)[\phi]_{\psi=0}$$

$$[M^2, B][\phi] := \left( \frac{\partial^2}{\partial\beta^2} + \frac{\partial^2}{\partial\mathbf{a}^2} \right) (T(\beta, \mathbf{a})B)[\phi]_{\beta=0, \mathbf{a}=0}$$

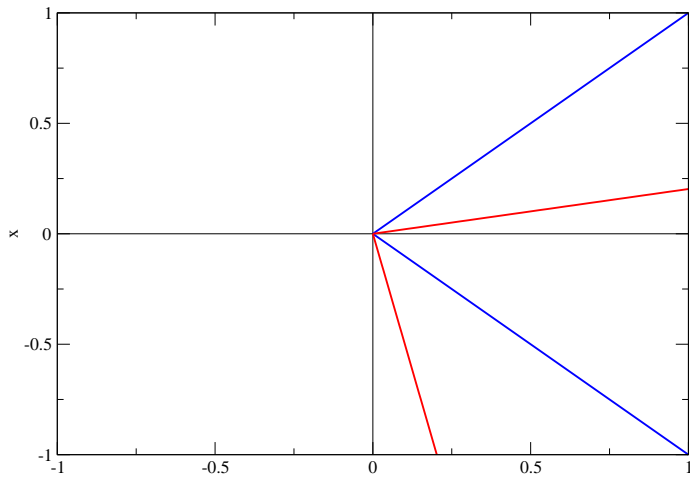
## One parameter groups and semigroups

$$T(\beta, \mathbf{a}) = e^{-\beta H + i\mathbf{a} \cdot \mathbf{P}}$$

$$U(R(\hat{\mathbf{n}}, \psi)) = e^{i\mathbf{J} \cdot \hat{\mathbf{n}}\psi}$$

$$W(\hat{\mathbf{n}}, \psi) = e^{\mathbf{K} \cdot \hat{\mathbf{n}}\psi}$$

**Domains for local symmetric semigroups**  
(A. Klein, L. Landau, J.F.A. 44(1981)121; Frörlich,  
Osterwalder, Seiler, Ann. Math 118(1983)461)



$$\{H, P, J, K\}$$

**Self-adjoint (on physical Hilbert space)**

$$H \geq 0 \quad (\text{Follows from reflection positivity})$$

**Satisfy Poincaré commutation relations**

**No analytic continuation used!**

**Given a reflection positive Euclidean Green function or generating function we have:**

- **Hilbert space scalar product.**
- **A dense set of normalizable vectors.**
- **A representation of the Poincaré Lie algebra in terms of self-adjoint operators.**

## Comments:

- While analytic continuation is not used, reflection positivity ensures the existence of an analytic continuation.
- We can exploit the ability to calculate matrix elements of all operators in a dense set of normalizable states.
- $e^{-\beta H}$  and  $H$  have the same eigenstates.

**Particles: mass eigenstates**

**Dense set + Gram-Schmidt**



**Orthonormal basis of “wave functionals”**

$$B_n[\phi] \quad \langle B_n | B_m \rangle = \delta_{mn}$$

**Solve for eigenstates in point spectrum of  $M^2$**

$$(M^2 B_\lambda)[\phi] = \lambda^2 B_\lambda[\phi]$$

$$B_\lambda[\phi] = \sum_n b_n B_n[\phi]$$

$$\sum_n \langle B_m | M^2 | B_n \rangle b_n = \lambda^2 b_m$$

**Particles: mass-momentum eigenstates  
(use translations and Fourier transforms)**

$$B_\lambda[\phi] = \sum_n b_n e^{i\phi(f_n)} \quad \text{(mass eigenfunctional)}$$

$$B_\lambda(\mathbf{p})[\phi] = \int \frac{d^3 a}{(2\pi)^{3/2}} e^{-i\mathbf{p}\cdot\mathbf{a}} [T(0, \mathbf{a}), B_\lambda][\phi]$$

$$\langle C | B_\lambda(\mathbf{p}) \rangle := \int \frac{d^3 a}{(2\pi)^{3/2}} e^{-i\mathbf{p}\cdot\mathbf{a}} \sum_{j=1}^{N_c} \sum_n b_n c_j^* Z[f_{n,\mathbf{a}} - \theta g_j]$$

**Particles: Mass-momentum-spin eigenstates  
(project on  $SU(2)$  irreducible representations)**

**Normalize  $B_\lambda(\mathbf{p})$**

$$B_\lambda(\mathbf{p})[\phi] \quad \langle B_\lambda(\mathbf{p}') | B_\lambda(\mathbf{p}) \rangle = \delta(\mathbf{p}' - \mathbf{p})$$

$$B_{\lambda,j}(\mathbf{p}, \mu)[\phi] := \int_{SU(2)} dR [U(R), B_\lambda(R^{-1}\mathbf{p})][\phi] D_{\mu j}^{j*}(R)$$

$$\langle C | B_{\lambda,j}(\mathbf{p}, \mu) \rangle :=$$

$$\int \frac{d^3 a dR}{(2\pi)^{3/2}} e^{-iR^{-1}\mathbf{p}\cdot\mathbf{a}} \sum_{j=1}^{N_c} \sum_n c_j^* b_n Z[f_{n,\mathbf{a},R} - \theta \mathbf{g}_j] D_{\mu j}^{j*}(R)$$

$$f_{n,\mathbf{a},R}(\tau, \mathbf{x}) = f_n(\tau, R\mathbf{x} - \mathbf{a})$$

## Finite Poincaré transformations of one-particle states ( $\lambda \in \sigma_{pp}$ )

**One-particle subspaces are irreducible subspaces with respect to the  
Poincaré group**

⇓

$$\langle C|U[\Lambda, a]|B_{\lambda,j}(\mathbf{p}, \mu)\rangle =$$

$$\sum_{\mu'=-j}^j \int d\mathbf{p}' \langle C|B_{\lambda,j}(\mathbf{p}', \mu')\rangle \mathcal{D}_{\mathbf{p}'\mu';\mathbf{p},\mu}^{\lambda,j}[\Lambda, a]$$

$$\mathcal{D}_{\mathbf{p}'\mu';\mathbf{p},\mu}^{\lambda,j}[\Lambda, a] =$$

$$\delta(\Lambda p - \mathbf{p}') \sqrt{\frac{\omega_{\lambda}(\mathbf{p}')}{\omega_{\lambda}(\mathbf{p})}} e^{-i\omega_{\lambda}(\mathbf{p}')a^0 - i\mathbf{p}' \cdot \mathbf{a}} D_{\mu'\mu}^j[\Lambda_c^{-1}(\frac{\mathbf{p}'}{\lambda}) \Lambda \Lambda_c(\frac{\mathbf{p}}{\lambda})]$$

$$\omega_{\lambda}(\mathbf{p}) = \sqrt{\lambda^2 + \mathbf{p} \cdot \mathbf{p}} \quad (\mathbf{p}')^j = \Lambda^j_{00} \omega_{\lambda}(\mathbf{p}) + \Lambda^j_k \mathbf{p}^k$$

**Here  $\Lambda_c(\frac{\mathbf{p}}{\lambda})$  is a rotationless Lorentz boost.**

## Scattering

- Time-dependent scattering has been used successfully to treat few-body problems (Kröger, Phys. Reports, 210(1992)46) in non-relativistic quantum mechanics.
- Calculations use wave packets and normalizable states.
- Haag-Ruelle scattering is a natural field theoretic generalization of non-relativistic time-dependent scattering theory (unlike LSZ it uses strong limits).

**Haag-Ruelle Scattering (P.R. 112(1958),668, Helv. Phys. Acta. 35(1962),147.)**

**Construct**

$$J : \otimes \mathcal{H}_{\lambda_i, j_i} = \mathcal{H}_f \rightarrow \mathcal{H} \quad U_f[\Lambda, a] = \otimes U_{\lambda_i, j_i}[\Lambda, a]$$

**The strong limit exists**

$$|\Psi_{\pm}\rangle = \lim_{t \rightarrow \pm\infty} e^{iHt} J e^{-iH_f t} |\Psi_{f\pm}\rangle = \Omega_{\pm} |\Psi_{f\pm}\rangle$$

**The wave operators satisfy (Ruelle H.P.A. 35(1962)34)**

$$U[\Lambda, a] \Omega_{\pm} = \Omega_{\pm} U_f[\Lambda, a]$$

## Structure of $J$

$$B_{\lambda,j}(\mathbf{p}, \mu)|0\rangle = |(\lambda, j)\mathbf{p}, \mu\rangle$$

**Creates one-particle state out of the vacuum**

$$\int J_i(\mathbf{p}_i, \mu_i) f(\mathbf{p}, \mu) d\mathbf{p} =$$
$$\int (-i\omega_\lambda(\mathbf{p}) B_{\lambda,j}(\mathbf{p}, \mu) - i[H, B_{\lambda,j}(\mathbf{p}, \mu)]) f(\mathbf{p}, \mu) d\mathbf{p}$$

( $\dots$ ) selects “creation part” of  $B_{\lambda,j}(\mathbf{p}, \mu)$

$$J(\mathbf{p}_1, \mu_1, \dots, \mathbf{p}_n, \mu_n) = \prod J_i(\mathbf{p}_i, \mu_i)|0\rangle$$

## Wave functional representation

$$B_{\lambda,j}(\mathbf{p}, \mu) \rightarrow B_{\lambda,j}(\mathbf{p}, \mu)[\phi]$$

$$\int J_i(\mathbf{p}_i, \mu_i)[\phi] f(\mathbf{p}, \mu) d\mathbf{p} =$$
$$\int (-i\omega_\lambda(\mathbf{p})B_{\lambda,j}(\mathbf{p}, \mu)[\phi] - i[H, B_{\lambda,j}(\mathbf{p}, \mu)][\phi]) f(\mathbf{p}, \mu) d\mathbf{p}$$

$$J[\phi] = \prod J_i(\mathbf{p}_i, \mu_i)[\phi]$$

## Two-space Scattering

$$\lim_{t \rightarrow \pm\infty} \|(e^{-iHt}|\Psi_{\pm}\rangle - J e^{-iH_f t}|\Psi_{f\pm}\rangle)\| = 0$$

$$|\Psi_{\pm}\rangle = \lim_{t \rightarrow \pm\infty} e^{iHt} J e^{-iH_f t} |\Psi_{f\pm}\rangle = \Omega_{\pm}(H, J, H_f) |\Psi_f\rangle$$

$$S_{fi} = \langle \Psi_+ | \Psi_- \rangle = \langle \Psi_{f+} | \Omega_+^{\dagger}(H, J, H_f) \Omega_-(H, J, H_f) | \Psi_{f-} \rangle$$

$$= \lim_{t \rightarrow \infty} \langle \Psi_{f+} | e^{iH_f t} J^{\dagger} e^{-2iHt} J e^{iH_f t} | \Psi_{f-} \rangle$$

## Kato-Birman invariance principle

$$\lim_{t \rightarrow \pm\infty} \|(e^{-iHt}|\Psi_{\pm}\rangle - Je^{-iH_f t}|\Psi_{f\pm}\rangle)\| = 0$$



$$\lim_{n \rightarrow \pm\infty} \|(e^{ine^{-\beta H}}|\Psi_{\pm}\rangle - Je^{ine^{-\beta H_f}}|\Psi_{f\pm}\rangle)\| = 0$$

Provides a possible computational strategy

$$\langle \Psi_{f+} | S | \Psi_{f-} \rangle \approx \langle \Psi_{f+} | e^{-ine^{-\beta H_f}} J^\dagger e^{2ine^{-\beta H}} J e^{-ine^{-\beta H_f}} | \Psi_{f-} \rangle$$

$$e^{2inx} \approx \sum_{m=0}^{N(n)} c_m(n) x^m \quad \rightarrow \quad e^{2ine^{-\beta H}} \approx \sum_{m=0}^{N(n)} c_m(n) e^{-m\beta H}$$

Convergence is **uniform** for each fixed  $n$ !

## Matrix elements

$$\langle B | e^{-m\beta H} | C \rangle = \sum_{j=1}^{N_b} \sum_{k=1}^{N_c} b_j^* c_k Z[g_{k,m\beta,0} - \theta f_j]$$

$$C[\phi] = J e^{-ine^{-\beta H_f}} |\Psi_{f-}\rangle[\phi]$$

$$B[\phi] = J e^{ine^{-\beta H_f}} |\Psi_{f-}\rangle[\phi]$$

**Computable by quadrature in terms of  $Z[f]$  or  $\{S_n\}$**

## Scattering in Euclidean space

- The results are **standard Minkowski space results** expressed in a representation where the Minkowski scalar product is evaluated in terms of a Euclidean generating functional.
- The time limits in the scattering theory are strong limits (compared with the weak limits used in LSZ scattering)
- The Haag-Ruelle scattering theory does not distinguish elementary and composite asymptotic states.
- Explicit representations of the Poincaré group exist for the bound and scattering states.

$$U[\Lambda, a]\Omega_{\pm} = \Omega_{\pm} U_f[\Lambda, a]$$

## Maiani-Testa No-Go Theorem (P.L.B. 245(1990)585)

$$\langle 0 | \phi_\pi(p_1) \phi_\pi(p_2) J(0) | 0 \rangle$$

- Use LSZ interpolating fields for pions. Field creates more than 1-pion states from vacuum.
- Uses  $\beta_1 \gg \beta_2 \gg 0$
- Uses Euclidean correlation functions.

### This approach

- Uses Haag-Ruelle fields. Fields create only 1 pion states from vacuum - products approach exact scattering states in strong limit.
- $\beta$  is a fixed adjustable parameter ( $H \leftrightarrow e^{-\beta H}$ ).
- Uses Minkowski scalar product.
- Calculations require wave packets, one-body solutions; no singularities, no analytic continuation.

**Summary of formal results**  
**Given a reflection positive generating functional**



- **Hilbert-space scalar product,  $\{H, P, J, K\}$**
- **Single-particle states**
- **Scattering states,  $S$ -matrix elements.**
- **Finite Poincaré transformations on single-particle states and scattering states.**

## Comments

- **Constructing a reflection positive Euclidean invariant generating functional is almost equivalent to constructing a non-trivial field theory (this must be relaxed for model applications).**
- **Practical calculations use a weakened form of reflection positivity (limited permutation symmetry).**
- **Full permutation symmetry = locality.**
- **Osterwalder-Schrader reconstruction of relativistic quantum mechanics does not require locality.**

## Green function approach - limited reflection positivity

$$Z[f] = \sum_n \frac{i^n}{n!} S_n(f, \dots, f)$$

$$x := (\tau, \mathbf{x}) \quad \theta x = (-\tau, \mathbf{x}) \quad f(x_1, \dots, x_n) \in \mathcal{S}_+$$

$$\int d^4x_1 \cdots d^4x_4 f_2^*(\theta x_2, \theta x_1) S_4(x_1, x_2; x_3, x_4) f_2(x_3, x_4) \geq 0$$

$$\int d^4x_1 d^4x_2 f_1^*(\theta x_1) S_2(x_1; x_2) f_1(x_2) \geq 0.$$

$$S_4(x_1, x_2; x_3, x_4) = S_4(x_2, x_1; x_3, x_4) = S_4(x_1, x_2; x_4, x_3)$$

## Green function representation

$$Z \rightarrow S = \begin{pmatrix} S_2(x; y) & S_3(x; y_1, y_2) & \cdots \\ S_3(x_1, x_2; y) & S_4(x_1, x_2; y_1, y_2) & \vdots \\ \vdots & \cdots & \ddots \end{pmatrix}$$

$$B[\phi] \rightarrow \begin{pmatrix} f_1(x_{11}) \\ f_2(x_{21}, x_{22}) \\ \vdots \end{pmatrix}$$

$$\langle B|C \rangle = (\theta f_B, S f_C)_e$$

$$\langle \mathbf{x} | \mathbf{H} | f \rangle := \{0, \frac{\partial}{\partial x_{11}^0} f_1(x_{11}), \left( \frac{\partial}{\partial x_{21}^0} + \frac{\partial}{\partial x_{22}^0} \right) f_2(x_{21}, x_{22}), \dots \}$$

$$\langle \mathbf{x} | \mathbf{P} | f \rangle := \{0, -i \frac{\partial}{\partial \bar{x}_{11}} f_1(x_{11}), -i \left( \frac{\partial}{\partial \bar{x}_{21}} + \frac{\partial}{\partial \bar{x}_{22}} \right) f_2(x_{21}, x_{22}), \dots \}$$

$$\langle \mathbf{x} | \mathbf{J} | f \rangle := \{0, -i \bar{x}_{11} \times \frac{\partial}{\partial \bar{x}_{11}} f_1(x_{11}), -i \left( \bar{x}_{21} \times \frac{\partial}{\partial \bar{x}_{21}} + \bar{x}_{22} \times \frac{\partial}{\partial \bar{x}_{22}} \right) f_2(x_{21}, x_{22}), \dots \}$$

$$\langle \mathbf{x} | \mathbf{K} | f \rangle := \{0, \left( \bar{x}_{11} \frac{\partial}{\partial x_{11}^0} - x_{11}^0 \frac{\partial}{\partial \bar{x}_{11}} \right) f_1(x_{11}), \left( \bar{x}_{21} \frac{\partial}{\partial x_{21}^0} - x_{21}^0 \frac{\partial}{\partial \bar{x}_{21}} + \bar{x}_{22} \frac{\partial}{\partial x_{22}^0} - x_{22}^0 \frac{\partial}{\partial \bar{x}_{22}} \right) f_2(x_{21}, x_{22}), \dots \}.$$

## Modifications for spin

$$\mathbf{J} : \quad \left( -i\vec{x}_{11} \times \frac{\partial}{\partial \vec{x}_{11}} \right) \rightarrow \left( -i\vec{x}_{11} \times \frac{\partial}{\partial \vec{x}_{11}} + \vec{\Sigma} \right)$$

$$\mathbf{K} : \quad \left( \vec{x}_{11} \frac{\partial}{\partial x_{11}^0} - x_{11}^0 \frac{\partial}{\partial \vec{x}_{11}} \right) \rightarrow \left( \vec{x}_{11} \frac{\partial}{\partial x_{11}^0} - x_{11}^0 \frac{\partial}{\partial \vec{x}_{11}} + \vec{B} \right)$$

where

$$\vec{\Sigma} = i\vec{\nabla}_\phi D(e^{\frac{-i}{2}\vec{\sigma}\cdot\vec{\phi}}, e^{\frac{i}{2}\vec{\sigma}^t\cdot\vec{\phi}})_{aa'}$$

and

$$\vec{B} = \vec{\nabla}_\rho D(e^{\frac{-i}{2}\vec{\sigma}\cdot\vec{\rho}}, e^{\frac{-i}{2}\vec{\sigma}^t\cdot\vec{\rho}})_{aa'}$$

and  $D(g_1, g_2)$  is a representation of  $SU(2) \times SU(2)$ .

## Two- body scattering

$$S_2(x; y) \quad K(x_1, x_2; y_2, y_1)$$

$$S_0 = S_2(x_1; y_1)S_2(x_2; y_2)$$

$$S_4 = S_0 + S_0KS_4$$

$$S = \begin{pmatrix} S_2(x; y) & 0 \\ 0 & S_4(x_1, x_2; y_1, y_2) \end{pmatrix}$$

$$\langle C|B \rangle = (\theta g_1, S_2 f_1)_e + (\theta g_2, S_4 f_2)_e$$

**Particles - scalar  
(case of free  $S_2$ )**

$$(f, \theta S_2 f)_e$$

$$\begin{aligned} &= \frac{1}{(2\pi)^4} \int d^4x d^4y d^4p f(x) \frac{e^{ip \cdot (\theta x - y)}}{p^2 + m^2} f(y) \\ &= \frac{1}{(2\pi)^4} \int d^4x d^4y d^4p f(x) \frac{e^{-ip_0 \cdot (x_0 + y_0) + i\vec{p} \cdot (\vec{x} - \vec{y})}}{(p^0 + i\omega_m(\vec{p}))(p^0 - i\omega_m(\vec{p}))} f(y) \\ &= \int d^3p \frac{|g(\vec{p})|^2}{2\omega_m(\vec{p})} \geq 0 \end{aligned}$$

where

$$g(\vec{p}) := \frac{1}{(2\pi)^{3/2}} \int d^4y f(y) e^{-\omega_m(\vec{p})y_0 - i\vec{p} \cdot \vec{y}}.$$

**Particles - fermions**  
**(Euclidean time reversal has a spinor component)**

$$\begin{aligned} & (f, \theta \gamma^0 S_2 f)_e \\ &= \frac{1}{(2\pi)^4} \int d^4x d^4y d^4p f(x) e^{i\mathbf{p} \cdot (\theta \mathbf{x} - y)} \gamma^0 \frac{m - \mathbf{p} \cdot \boldsymbol{\gamma}_e}{p^2 + m^2} f(y) \\ &= \int g^\dagger(\vec{p}) \frac{\Lambda_+(p)}{(2\pi)^3} g(\vec{p}) d^3p \end{aligned}$$

where

$$\Lambda_+(p) := \frac{\omega_m(\vec{p}) + \gamma^0 \vec{\gamma} \cdot \vec{p} - m\gamma^0}{2\omega_m(\vec{p})}$$

## Scattering in Euclidean space

**Approximation 1:** Use sharply peaked (in momentum) normalizable states to approximate plane-wave on-shell transition matrix elements.

$$\langle \Psi_{f+} | S | \Psi_{f-} \rangle = \langle \Psi_{f+} | \Psi_{f-} \rangle - 2\pi i \langle \Psi_{f+} | \delta(E_+ - E_-) T | \Psi_{f-} \rangle$$

$$\langle \mathbf{p}'_1, \mu'_1, \mathbf{p}'_2, \mu'_2 | T | \mathbf{p}_1, \mu_1, \mathbf{p}_2, \mu_2 \rangle \approx \frac{\langle \Psi_{f+} | S | \Psi_{f-} \rangle - \delta_{ab} \langle \Psi_{f+} | \Psi_{f-} \rangle}{2\pi i \langle \Psi_{f+} | \delta(E_+ - E_-) | \Psi_{f-} \rangle}$$

## Scattering injection operators (N=2)

**Approximation 2:** Calculate  $\psi_\lambda(\mathbf{x}, \tau; \mathbf{p}, \mu)$

- $\psi_\lambda(\mathbf{x}, \tau; \mathbf{p}, \mu)$ : eigenstate of  $M^2, \mathbf{P}, j^2, j_z$  in one-body Hilbert space generated by  $S_2$ .

$$J_i(\mathbf{x}, \tau; \mathbf{p}, \mu) := (-i\omega_\lambda(\mathbf{p}) + i\frac{\partial}{\partial\tau})\psi_\lambda(\mathbf{x}, \tau; \mathbf{p}, \mu)$$

$$J(\mathbf{x}_1, \tau_1, \mathbf{x}_2, \tau_2; \mathbf{p}_1, \mu_1, \mathbf{p}_2, \mu_2) = J_1(\mathbf{x}_1, \tau_1; \mathbf{p}_1, \mu_1)J_2(\mathbf{x}_2, \tau_2; \mathbf{p}_2, \mu_2)$$

## Scattering in Euclidean space

Use time-dependent scattering to calculate  $S$  matrix elements in normalizable states.

Use Kato-Birman invariance principle to express  $S$  in terms of  $e^{-\beta H}$ .

$$\langle \Psi_{f+} | S | \Psi_{f-} \rangle$$

$$= \lim_{t \rightarrow \infty} \langle \Psi_{f+} | e^{iH_f t} J^\dagger e^{-2iHt} J e^{iH_f t} | \Psi_{f-} \rangle$$

$$= \lim_{n \rightarrow \infty} \langle \Psi_{f+} | e^{-ine^{-\beta H_f}} J^\dagger e^{2ine^{-\beta H}} J e^{-ine^{-\beta H_f}} | \Psi_{f-} \rangle$$

## Scattering in Euclidean space

**Approximation 3:** Replace  $n$  by large fixed  $n$ .

$$\begin{aligned} & \langle \Psi_{f+} | S | \Psi_{f-} \rangle \\ \approx & \langle \Psi_{f+} | e^{-ine^{-\beta H_f}} J^\dagger e^{2ine^{-\beta H}} J e^{-ine^{-\beta H_f}} | \Psi_{f-} \rangle \end{aligned}$$

## Approximation 4: Uniform polynomial approximation

$$e^{2ine^{-\beta H}} \approx \sum c_m(n)(e^{-\beta m H})$$

note  $\sigma(e^{-\beta H}) \in [0, 1]$  (**compact**)

$$e^{2inx} \approx \sum c_m(n)x^m \quad x \rightarrow e^{-\beta H}$$

$$|e^{2inx} - \sum c_m(n)x^m| < \epsilon(n) \quad \forall x \in [0, 1]$$

↓

$$\| [e^{2ine^{-\beta H}} - \sum c_m(n)(e^{-\beta m H})] |\psi\rangle \| < \epsilon(n) \| |\psi\rangle \|$$

$$f(x) \approx \frac{1}{2} c_0 T_0(x) + \sum_{k=1}^N c_k T_k(x)$$

$$c_j = \frac{2}{N+1} \sum_{k=1}^N f\left(\cos\left(\frac{2k-1}{N+1} \frac{\pi}{2}\right)\right) \cos\left(j \frac{2k-1}{N+1} \frac{\pi}{2}\right)$$

$$f(e^{-\beta H}) \approx \frac{1}{2} c_0 T_0(e^{-\beta H}) + \sum_{k=1}^N c_k T_k(e^{-\beta H})$$

$$f(x) = e^{2inx}$$

$$|e^{2inx} - P_N(x)| < 2 \frac{n^{N+1}}{(N+1)!}$$

$$\langle \Psi_{f+} | S | \Psi_{f-} \rangle \approx$$

$$= \sum c_m(n) \langle \Psi_{f+} | e^{-ine^{-\beta H_f}} J^\dagger(e^{-\beta m H}) J e^{-ine^{-\beta H_f}} | \Psi_{f-} \rangle$$

**Each approximation converges - the order of the approximations is important (1)  $\rightarrow$  (2)  $\rightarrow$  (3)  $\rightarrow$  (4).**

## Three and four-body phenomenology and cluster properties

$$S^{-1}(123) = \\ S^{-1}(1)S^{-1}(2)S^{-1}(3) - K(12)S^{-1}(3) - \\ K(23)S^{-1}(1) - K(31)S^{-1}(2) - K(123)$$

$$S^{-1}(1234) = \\ S^{-1}(1)S^{-1}(2)S^{-1}(3)S^{-1}(4) + K(12)K(34) + K(13)K(24) + \\ K(14)K(23) - K(12)S^{-1}(3)S^{-1}(4) - K(13)S^{-1}(2)S^{-1}(4) + \\ -K(14)S^{-1}(2)S^{-1}(3) - K(23)S^{-1}(1)S^{-1}(4) + \\ -K(24)S^{-1}(1)S^{-1}(3) - K(34)S^{-1}(1)S^{-1}(2) + \dots$$

**Test of method: non-relativistic separable potential  
(solvable so all approximations can be tested)**

$$H = \frac{\mathbf{k}^2}{m} - |g\rangle\lambda\langle g|$$

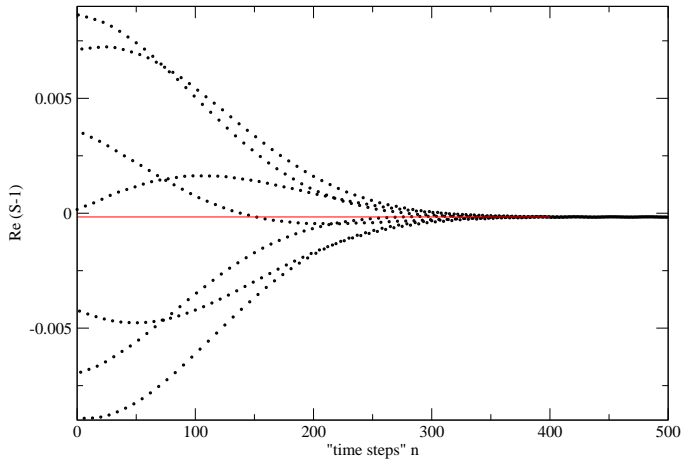
$$\langle \mathbf{k}|g\rangle = \frac{1}{m_\pi^2 + \mathbf{k}^2}$$

**calculate  $\langle \mathbf{k}'|T(k^+)|\mathbf{k}\rangle$  using matrix elements of  $e^{-\beta H}$  in normalizable states.**



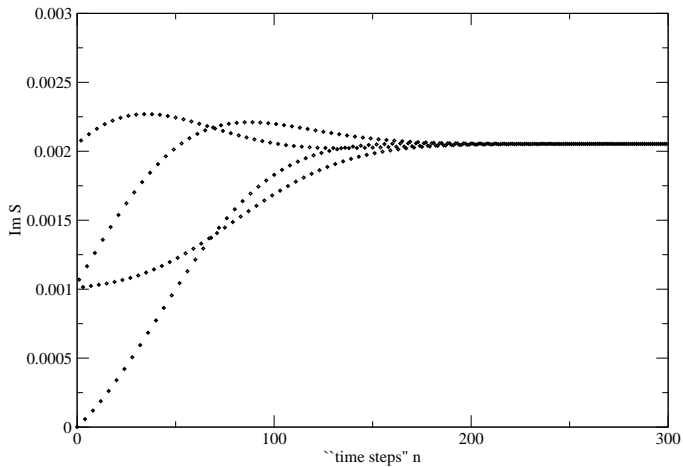
### Approximation 3:

Real S-1 vs n-limit - 1 GeV



### Approximation 3:

Im S vs n-limit - 2.1 GeV

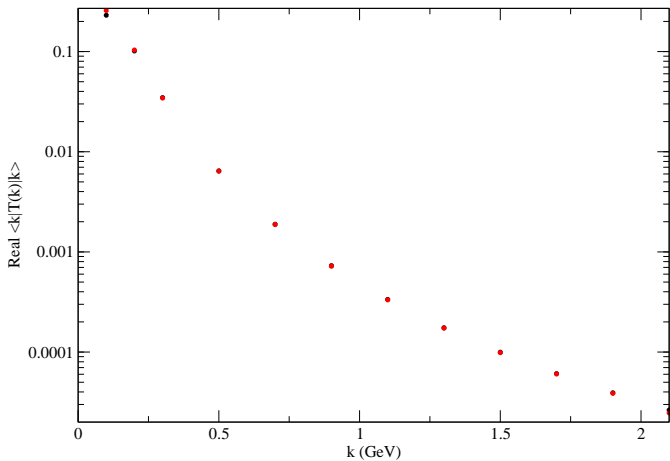


### Approximation 4:

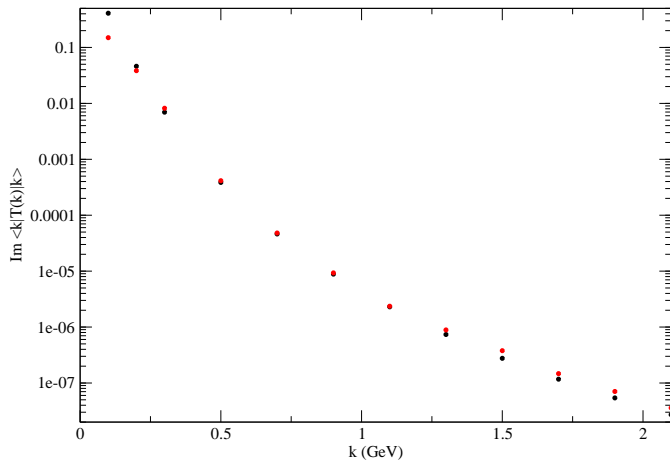
Degree 300 polynomial compared to  $e^{-inx}$ ,  $n = 220$

$x$	$\Delta \cos(nx)$	$\Delta \sin(nx)$
0	$4.44089 \times 10^{-16}$	$8.32667 \times 10^{-15}$
0.1	$2.35367 \times 10^{-14}$	$1.46966 \times 10^{-14}$
0.2	$5.55112 \times 10^{-16}$	$3.6797 \times 10^{-14}$
0.3	$3.84137 \times 10^{-14}$	$1.80689 \times 10^{-14}$
0.4	$1.72085 \times 10^{-14}$	$1.32672 \times 10^{-14}$
0.5	$2.77556 \times 10^{-15}$	$2.93793 \times 10^{-14}$
0.6	$6.66134 \times 10^{-16}$	$3.33344 \times 10^{-14}$
0.7	$8.54872 \times 10^{-15}$	$2.50355 \times 10^{-14}$
0.8	$1.02141 \times 10^{-14}$	$1.35447 \times 10^{-14}$
0.9	$1.22125 \times 10^{-15}$	$2.72282 \times 10^{-14}$
1	$4.88498 \times 10^{-15}$	$6.61415 \times 10^{-14}$

Real part of  $\langle k|T(k)|k\rangle$  (exact - black, polynomial - red)



Im part of  $\langle k|T(k)|k\rangle$  (exact - black, polynomial - red)



## Conclusions - Outlook

- Phenomenology based on model reflection-positive Euclidean Green functions can be used to formulate a relativistic quantum theory.
- Analytic continuation is not necessary.
- The Poincaré invariant S-matrix. Cluster properties are easily satisfied for fixed  $N$ .
- Models can be motivated by field-theory based phenomenology.
- A test using an exactly solvable model suggests that GeV scale scattering calculations are possible in this framework.

## Future directions

- **Euclidean BS free  $S_2$ .**
- **Euclidean BS  $S_2$  with continuous Lehmann weight.**
- **Nakanishi representation and reflection positivity.**
- **Current matrix elements.**

**Thanks!**

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