

Wavelet Bases in Few-Body Scattering Calculations

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Outline

- **Background**
- **Scaling functions**
- **Multiscale analysis and wavelets**
- **Wavelet numerical analysis**
- **Applications**

What are wavelets?

They are orthonormal basis functions that are used in data compression algorithms.

JPEG digital images are tables of expansion coefficients in a wavelet basis.

FBI fingerprint files are stored as expansion coefficients in a wavelet basis.

Our interest in wavelets?

Scattering problems in Poincaré invariant quantum mechanics can be formulated as integral equations with compact kernels.

Compact operators can be uniformly approximated by finite-dimensional matrices.

Large matrices can be uniformly approximated by **sparse matrices** in a wavelet basis.

Wavelet bases can be used to improve the efficiency of **momentum-space** scattering calculations.

Interesting considerations

Wavelet bases are natural for problems (quantum field theory, phase transitions) where **many scales are strongly coupled**.

Basis functions are related to fixed points of a linear **renormalization group** equation.

Basis functions have a **fractal structure**, and are not amenable to standard numerical methods.

Elements of wavelet numerical analysis

Unitary operators, D and T on $L^2(\mathbb{R})$:

$$D(f)(x) := \sqrt{\frac{1}{2}} f\left(\frac{x}{2}\right) \quad T(f)(x) := f(x-1)$$

Scaling equation:

$$D(\phi)(x) = \sum_{l=0}^{2k-1} h_l T^l(\phi)(x). \quad (1)$$

The scaling function, $\phi(x)$, is a fixed point of (1) with normalization

$$\int \phi(x) dx = 1.$$

h_l are constant coefficients that determine the type of wavelet (we use Daubechies' wavelets).

k is a finite integer.

A necessary condition for the existence of a fixed point of the scaling equation is:

$$\sum_{l=1}^{2k-1} h_l = \sqrt{2}.$$

Daubechies' scaling coefficients, $k = 1, 2, 3$

h_l	k=1	k=2	k=3
h_0	$1/\sqrt{2}$	$(1 + \sqrt{3})/4\sqrt{2}$	$(1 + \sqrt{10} + \sqrt{5 + 2\sqrt{10}})/16\sqrt{2}$
h_1	$1/\sqrt{2}$	$(3 + \sqrt{3})/4\sqrt{2}$	$(5 + \sqrt{10} + 3\sqrt{5 + 2\sqrt{10}})/16\sqrt{2}$
h_2	0	$(3 - \sqrt{3})/4\sqrt{2}$	$(10 - 2\sqrt{10} + 2\sqrt{5 + 2\sqrt{10}})/16\sqrt{2}$
h_3	0	$(1 - \sqrt{3})/4\sqrt{2}$	$(10 - 2\sqrt{10} - 2\sqrt{5 + 2\sqrt{10}})/16\sqrt{2}$
h_4	0	0	$(5 + \sqrt{10} - 3\sqrt{5 + 2\sqrt{10}})/16\sqrt{2}$
h_5	0	0	$(1 + \sqrt{10} - \sqrt{5 + 2\sqrt{10}})/16\sqrt{2}$

Computing $\phi(x)$

Theorem: $\text{support}[\phi(x)] \in [0, 2k - 1]$

Theorem: $k > 1 \Rightarrow \phi(x)$ continuous.

$$\phi(n) = \sqrt{2} \sum_{l=0}^{2k-1} \sqrt{2} h_l \phi(2n - l).$$

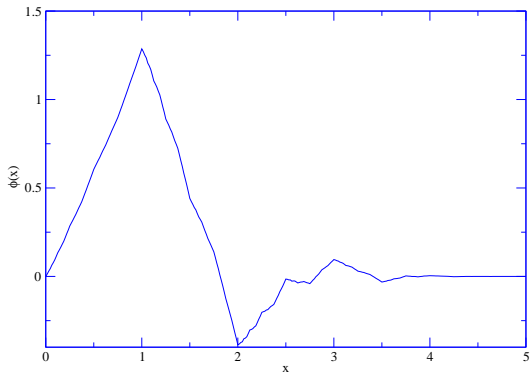
$$\sum_n \phi(n) = 1.$$

Solving above equations gives $\phi(n)$, $n \in [0, 2k - 1]$.

$$\phi\left(\frac{n}{2}\right) = \sum_{l=0}^{2k-1} \sqrt{2} h_l \phi(n - l).$$

Induction gives exact values at all dyadic rationals.

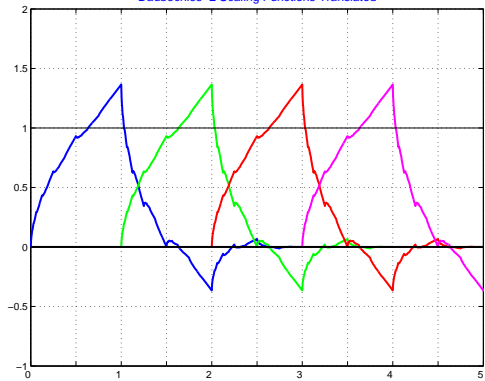
Daubechies' K=3 scaling function



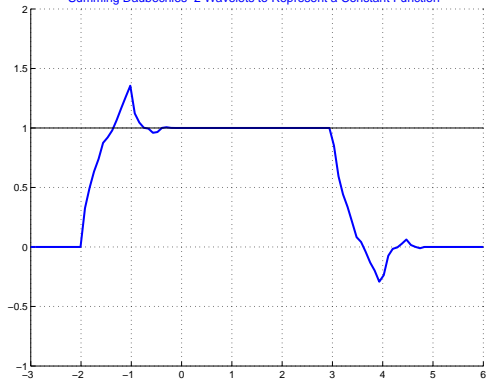
Approximation spaces

- $\phi_{mn}(x) := D^m T^n \phi(x)$.
- $(\phi_{mn}, \phi_{mk}) = \delta_{nk}$ (fixed m).
- $\{\phi_{mn}\}_n$ can **locally pointwise** represent polynomials of degree $< k$ for each m .
- $\mathcal{V}_m := \text{span}\{\phi_{mn}\}_n \cap L^2(\mathbb{R}) =$ “**scale m subspace**” of $L^2(\mathbb{R})$.

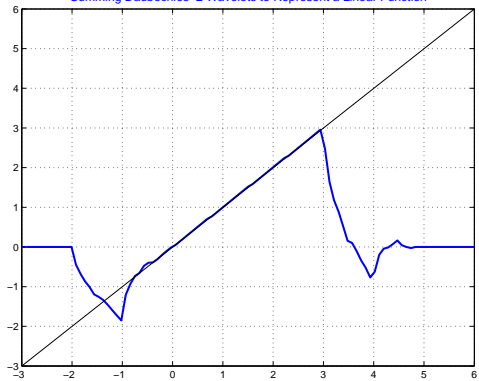
Daubechies-2 Scaling Functions Translated



Summing Daubechies-2 Wavelets to Represent a Constant Function



Summing Daubechies-2 Wavelets to Represent a Linear Function



Multiscale analysis

$$m > n \Rightarrow \mathcal{V}_n \supset \mathcal{V}_m$$

$$L^2(\mathbb{R}) \supset \cdots \supset \mathcal{V}_{n-1} \supset \mathcal{V}_n \supset \mathcal{V}_{n+1} \supset \cdots \supset \emptyset$$

Theorem: $\lim_{n \rightarrow -\infty} \mathcal{V}_n = L^2(\mathbb{R})$

$$\mathcal{V}_n = \mathcal{V}_{n+1} \oplus \mathcal{W}_{n+1}$$

\Downarrow

$$\mathcal{V}_n = \mathcal{W}_{n+1} \oplus \mathcal{W}_{n+2} \oplus \cdots \oplus \mathcal{W}_{n+m} \oplus \mathcal{V}_{n+m}$$

$$L^2(\mathbb{R}) = \bigoplus_{n=-\infty}^{\infty} \mathcal{W}_n = \mathcal{V}_m \oplus \left(\bigoplus_{n=-\infty}^m \mathcal{W}_n \right)$$

Wavelets

\mathcal{W}_n are **wavelet** spaces

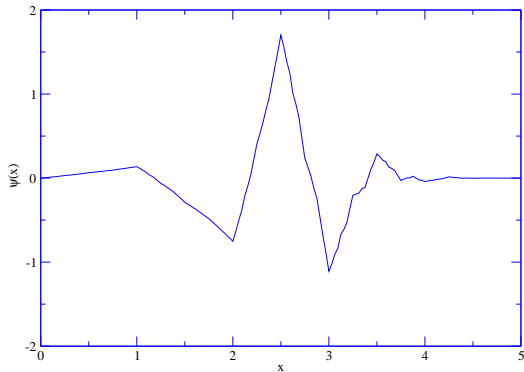
$$D\psi(x) = \sum_{l=0}^{2k-1} (-)^l h_{2k-l-1} T^l \phi(x)$$

$$\psi_{ml}(x) = D^m T^l \psi(x)$$

$\{\psi_{ml}\}_l$ orthonormal basis for \mathcal{W}_m

$\psi(x)$ is called the “**Mother**” wavelet

Daubechies' $k = 3$ mother wavelet



$$\text{support } [\psi(x)] = \text{support } [\phi(x)]$$

h_l are determined up to space reflection by the requirements

$$(\psi, x^n) = 0, \quad n = 0, \dots, k-1 \quad (\phi, T^m \phi) = \delta_{m0}$$

\Downarrow

$$\sum_m m^n (-)^m h_{l-m} = 0 \quad \sum_{l=0}^{2k-1} h_{l-2m} h_l = \delta_{m0}$$

$$\sum_{l=1}^{2k-1} h_l = \sqrt{2}$$

Properties

$\phi_m(x)$ can locally **pointwise represent** polynomials of degree $< k$.

$\psi_{mn}(x)$ are **orthogonal** to polynomials of degree $< k$.

$$\mathcal{V}_n = \mathcal{W}_{n+1} \oplus \mathcal{W}_{n+2} \oplus \cdots \oplus \mathcal{W}_{n+l} \oplus \mathcal{V}_{n+l}$$

Approximations

Fix scale n of approximate solution

$$\mathcal{V}_n = \mathcal{W}_{n+1} \oplus \mathcal{W}_{n+2} \oplus \cdots \oplus \mathcal{W}_{n+m} \oplus \mathcal{V}_{n+m}$$

$$W : \mathcal{V}_n \rightarrow \mathcal{W}_{n+1} \oplus \mathcal{W}_{n+2} \oplus \cdots \oplus \mathcal{W}_{n+m} \oplus \mathcal{V}_{n+m}$$

Wavelet transform W ($O(N)$ orthogonal transformation)

$$f(x) = \sum_m a_m \phi_{nm}(x) = \sum_m a'_m \phi_{n+l,m}(x) + \sum_m \sum_{k=1}^l b_{n+k,m} \psi_{n+k,m}(x)$$

Sparse matrices

$$a_n \underbrace{\leftrightarrow}_{W} a'_n, b_{nm}$$

b_{nm} **small** if $f(x)$ can be well approximated by degree $k - 1$ polynomial on the support of $\psi_{nm}(x)$.

$$\text{supp}(\psi_{nm}(x)) \subseteq [2^n m, 2^n(m + 2k - 1)]$$

$$WMW^T = M_1 + M_2 \quad \|M_2\| < \epsilon, M_1 \text{ sparse}$$

Wavelet numerical analysis

Moments

$$\langle x^0 \rangle_\phi := (x^0, \phi) = 1$$

$$\langle x^m \rangle_\phi := (x^m, \phi) = (Dx^m, D\phi) = 2^{-m-1/2} \sum_l h_l(x^m, T^l \phi) \Rightarrow$$

$$\langle x^m \rangle_\phi = \frac{1}{\sqrt{2}} \frac{1}{2^m - 1} \sum_{l=0}^{2k-1} \sum_{n=0}^{m-1} h_l l^{m-n} \frac{m!}{n!(m-n)!} \langle x^n \rangle_\phi$$

recursion



$$\langle x^m \rangle_\phi, \quad \langle x^n \rangle_{\phi_{lm}}, \quad \langle x^n \rangle_{\psi_{lm}} \quad \text{exact}$$

Low-degree polynomials

$$x^m = \sum_n c_n^m \phi_n(x)$$

exact for $m < k$

$$c_n^m = (T^n \phi, x^m) = \sum_{l=0}^m \frac{m! n^{m-l}}{l!(m-l)!} \langle x^l \rangle_\phi$$

$$c_n^0 = 1; \quad c_n^1 = n + \langle x^1 \rangle_\phi, \quad \dots$$

Smoothness

$$\phi(x) = \sqrt{2} \sum_{l=0}^{2k-1} h_l \phi(2x - l)$$

↓

$$\phi(n) = \sqrt{2} \sum_m H_{nm} \phi(m) \quad H_{mn} := h_{2n-m}$$

$$\frac{d^l \phi}{dx^l}(n) = 2^l \sqrt{2} \sum_m H_{nm} \frac{d^l \phi}{dx^l}(m)$$

ϕ has l derivatives if H_{mn} has an eigenvalue $\lambda = 2^l \sqrt{2}$

Derivatives

$$x = \sum_n c_n^1 \phi_n(x) \Rightarrow 1 = \sum_n c_n^1 \frac{d\phi_n}{dx}(x) \quad c_n^1 = n + \langle x^1 \rangle_\phi$$

$$D\left(\frac{d\phi}{dx}\right)(x) = 2 \sum_{l=0}^{2k-1} h_l T^l\left(\frac{d\phi}{dx}\right)(x).$$

⇓

$$\frac{d\phi}{dx}(x) \quad \left(\frac{d\phi}{dx}, \phi_n\right),$$

$$\frac{d\phi}{dx}(x) \approx \sum_n \left(\frac{d\phi}{dx}, \phi_n\right) \phi_n(x)$$

Derivatives

$$\frac{d\phi_{mn}}{dx}(x) \quad \frac{d\psi_{mn}}{dx}(x)$$

can be calculated **exactly** at dyadic rationals.

$$\left(\phi_{mn}, \frac{d\phi_{mk}}{dx}\right) \quad \left(\psi_{nl}, \frac{d\phi_{mk}}{dx}\right) \quad \left(\phi_{mn}, \frac{d\psi_{mk}}{dx}\right) \quad \left(\psi_{nl}, \frac{d\psi_{mk}}{dx}\right)$$

can all be computed **exactly**.

Nonlinearities

$$\phi_{n_1}(x)\phi_{n_2}(x) \approx \sum_{n_3} \Gamma_{n_1, n_2}^{n_3} \phi_{n_3}(x)$$

$$\Gamma_{n_1, n_2}^{n_3} = \int \phi_{n_1}(x)\phi_{n_2}(x)\phi_{n_3}(x)dx =$$

$$\sum_{l_1, l_2, l_3} \sqrt{2} h_{l_1} h_{l_2} h_{l_3} \Gamma_{2n_1+l_1, 2n_2+l_2}^{2n_3+l_3}$$

$$\sum_{n_3} \Gamma_{n_1, n_2}^{n_3} = \delta_{n_1, n_2} \quad \Gamma_{n_1, n_2}^{n_3} = \Gamma_{n_1-n_3, n_2-n_3}^0$$



$$\Gamma_{n_1, n_2}^{n_3}, \dots$$

Boundary integrals

$$B_m = \int_m^\infty \phi(x) dx = \frac{1}{2} \sum_{l=1}^{2k-1} h_l B_{2m-l}$$

$$B_m = 0 \quad m \geq 2k - 1 \quad B_m = 1 \quad m \leq 0$$

$$B^m = \int_{-\infty}^m \phi(x) dx = 1 - B_m$$

Poles

$$J_n := \left(\phi_n, \frac{1}{x - i0^+} \right) = \left(D\phi_n(x), D\frac{1}{x - i0^+} \right) =$$
$$\sqrt{2} \sum_{l=0}^{2k-1} h_l(\phi_{2n+l}, \frac{1}{x}) = \sqrt{2} \sum_{l=0}^{2k-1} h_l J_{2n+l}$$

large n

$$J_n = \frac{1}{n} \sum_{m=0}^{\infty} \frac{\langle x^m \rangle_{\phi}}{n^m}$$

$$i\pi = \int_{-m}^m \frac{dx}{x - i0^+} = \sum_n J_n + \text{boundary terms}$$

Logarithmic singularities

$$L_n := (\phi_n, \ln) = (D\phi_n, D\ln) =$$

$$\frac{1}{\sqrt{2}} \sum_{l=0}^{2k-1} h_l L_{2n-l} - \ln(2)$$

large n

$$L_n = \ln(n) - \sum_{m=1}^{\infty} (-1)^m \frac{\langle X^m \rangle_{\phi}}{mn^m}$$

One-point quadrature

$$k > 1 \Rightarrow \langle x^n \rangle_\phi = \langle x \rangle_\phi^n, \quad n = 0, 1, 2$$

↓

$$\int \phi(x) P(x) dx \approx P(\langle x \rangle_\phi)$$

exact for $P(x)$ a polynomial of degree 2!

Moving singularities

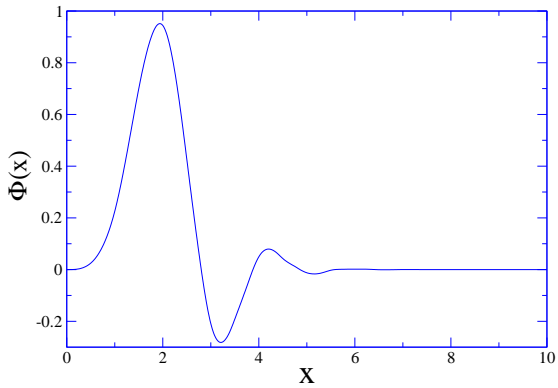
$$J_{mn} := \int \frac{\phi_m(x)\phi_n(y)}{k-x-y} dx dy =$$

$$\int \frac{\Phi(x)}{k-x-m-n} dx$$

$$\Phi(x) := \int \phi(y)\phi(x-y) dy$$

$$D\Phi(x) = \sum_l r_l T^l \Phi(x) \quad \int \Phi(x) dx = 1$$

Autocorrelation function



$$f(x) = g(x) + \int \frac{K(x, y)}{y - i0^+} f(y) dy$$

$$f \approx \sum f_n \phi_{mn}(x)$$

$$f_l = g(x_l) + \sum_{nk} K(x_l, y_n) \Gamma_{kn}^l J_k w f_n$$

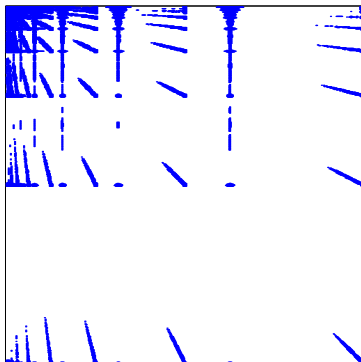
$$\mathcal{W} \Rightarrow \text{solve} \Rightarrow \mathcal{W}^{-1} \Rightarrow$$

$$f(x) = g(x) + \sum_{nk} K(x, y_n) \Gamma_{kn}^l J_k w f_n$$

$$J_n := 2^{m/2} \int dx \frac{\phi_{mn}(x)}{x - i0^+}$$

Required input

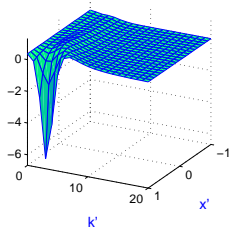
- a. **Kernel at one point quadrature points**
- b. **Driving at one point quadrature points**
- c. **Integrals Γ_{mn}^l and J_k on one scale**



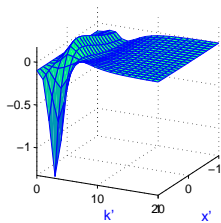
K=3, E=10 MeV, J=-7

ϵ	percent	on-shell value	on-shell error	mean-square error
0	100	-125.00480	0	0
10^{-9}	17.78	-125.00480	1.05×10^{-8}	2.56×10^{-8}
10^{-8}	11.38	-125.00480	5.14×10^{-8}	2.44×10^{-7}
10^{-7}	6.6	-125.00475	4.49×10^{-7}	1.88×10^{-6}
10^{-6}	3.76	-125.00269	1.69×10^{-5}	2.08×10^{-5}
10^{-5}	2.14	-124.99030	.000116	.000228
10^{-4}	1.24	-124.85112	.00123	.00217
10^{-3}	.72	-123.82508	.00944	.0117
10^{-2}	.38	-125.25766	.00202	.128

Re(T)



Im(T)



Quantum fields $\Phi(x)$

$$\Phi(x) = \sum_{n=-\infty}^{\infty} \Phi(m, n) \phi_{m,n}(x) + \sum_{k=-\infty}^{m-1} \sum_{n=-\infty}^{\infty} \Psi(m, n) \psi_{m,n}(x)$$

$$\Phi(m, n) := \int \phi_{m,n}(x) \Phi(x) dx$$

$$\Psi(m, n) := \int \psi_{m,n}(x) \Phi(x) dx$$

Volume Cutoff N (both sums)

$$\sum_{n=-\infty}^{\infty} \rightarrow \sum_{n=-N}^N$$

Resolution Cutoff M (wavelet field sum)

$$\sum_{k=-\infty}^{m-1} \rightarrow \sum_{k=M}^{m-1}$$

$$\frac{d\Phi}{dx}(x) = \sum_{n=-\infty}^{\infty} \Phi(m, n) \frac{d\phi_{m,n}}{dx}(x) + \sum_{k=-\infty}^{m-1} \sum_{n=-\infty}^{\infty} \Psi(m, n) \frac{d\psi_{m,n}}{dx}(x)$$

Derivative of the wavelet or scaling function can be expanded in basis of wavelets and scaling functions

$$\begin{aligned} \frac{d\phi_{m,n}}{dx}(x) = & \\ & \sum_{n=-\infty}^{\infty} \left(\frac{d\phi_{m,n}}{dx}, \phi_{m,n} \right) \phi_{m,n}(x) + \\ & \sum_{k=-\infty}^{m-1} \sum_{n=-\infty}^{\infty} \left(\frac{d\phi_{m,n}}{dx}, \psi_{k,n} \right) \psi_{k,n}(x) \end{aligned}$$

⋮

Operator products

$$\Phi(x)^2 =$$

$$\sum_{n_1, n_2 = -\infty}^{\infty} \Phi(m, n_1) \Phi(m, n_2) \phi_{m, n_1}(x) \phi_{m, n_2}(x) +$$

$$\sum_{n_1, n_2 = -\infty}^{\infty} \sum_{k = -\infty}^{m-1} \Phi(m, n_1) \Psi(k, n_2) \phi_{m, n_1}(x) \psi_{k, n_2}(x) +$$

$$\sum_{n_1, n_2 = -\infty}^{\infty} \sum_{k = -\infty}^{m-1} \Psi(k, n_1) \Phi(m, n_2) \psi_{k, n_1}(x) \phi_{m, n_2}(x) +$$

$$\sum_{n_1, n_2 = -\infty}^{\infty} \sum_{k_1, k_2 = -\infty}^{m-1} \Psi(k_1, n_1) \Psi(k_2, n_2) \psi_{m, n_1}(x) \psi_{m, n_2}(x)$$

$$\phi_{m,n_1}(x)\phi_{m,n_2}(x) =$$

$$\sum_{n_3=-\infty}^{\infty} \phi_{m,n_3}(x) \int \phi_{m,n_3}(y)\phi_{m,n_1}(y)\phi_{m,n_2}(y)dy$$

$$\sum_{n_3=-\infty}^{\infty} \sum_{k=-\infty}^{m-1} \psi_{k,n_3}(x) \int \psi_{k,n_3}(y)\phi_{m,n_1}(y)\phi_{m,n_2}(y)dy$$

⋮

Change of scale

$$\Phi(m, n) = W_{nl}^{m1} \Phi(m-1, l) + W_{nl}^{m2} \Psi(m-1, l)$$

W_{nl}^{m1} and W_{nl}^{m2} exactly computable

Use resolution cutoff **after** computing all derivatives and operator products. Cutoff gives projection on subspace of given resolution.

$$[G_i(x), G_j(y)] = ig_{ijk}\delta(x - y)G_k(x)$$

$$\sum_n \psi_n(x) = 1 \quad (\text{partition of unity})$$

$$\sum_{mn} [G_i(n), G_j(m)] = ig_{ijk} \sum_n G_j(n)$$

$$G_i(n) = \int \phi_n(x) G_i(x) dx$$

- **Generalization of block-spin approximation. Has more smoothness.**
- **Map connecting different resolutions is invertible.**
- **Scale invariance means that the dependence on the Ψ fields vanishes.**
- **Local operator products and derivatives of fields are described in terms of analytically computable overlap integrals.**
- **Ψ fields describe physics lost on changing scales.**
- **Natural projections on given scale.**

Open Questions?

- Reflection Positivity?
- Exact local gauge invariance?
- Poincare invariance?

Conclusion

- Wavelet bases have all of the advantages of spline bases with the additional properties:
 - a. orthonormal basis.
 - b. sparse matrix (even in momentum space).
 - c. wavelet transform automatically finds structure.
 - d. application to scattering with Malfliet-Tjon potential more accurate than what we can obtain with spline basis.
- Tested on two-body scattering using partial wave and direct three-dimensional integration.
- Interesting possibilities for discrete field theories.