Cluster Properties and Relativistic Quantum Mechanics

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Why is quantum field theory difficult?

- ∞ number of degrees of freedom.
- No rigorous ab-initio error bounds.
- Fock-space generators not generally densely defined

$$H = \int d^3x T^{00}(x) \qquad \langle\psi|H^2|\psi\rangle = \infty$$

regularization destroys commutation relations.

- No few-body problem.

Why is non-relativistic few-body quantum mechanics successful ?

- Meaningful few-body problem directly related to experiment.
- Solutions with ab-initio error bounds possible.
- Cluster properties relate the few- and many-body problem.

Relativistic few-body quantum mechanics ?

- Determine degrees of freedom by experiment.
- Construct the most general class quantum models with desired degrees of freedom and exact Poincaré symmetry.
- Constrain few-body dynamics by experiment.
- Use cluster properties to relate few and many-body problem.

It is possible?

- (Weinberg) “There have been many attempts to formulate a relativistically invariant theory that would not be a local field theory, and it is indeed possible to construct theories that are not field theories and yet yield a Lorentz invariant S-matrix for two- particle scattering, but such efforts have always run into trouble in sectors with more than two particles; either the three particle S-matrix is not Lorentz invariant, or else it violates the cluster decomposition principle.”
- (Schroer) *An anthology of non-local QFT and QFT on noncommutative spacetime*, Bert Schroer, hep-th/0405105 “ This raised the question whether there exist consistent relativistic unitary and macro-causal particle theories are all. $\cdots$ In particular these theories fulfill the very non-trivial cluster separability properties of the associated Poincaré invariant unitary S-matrix; as a result their existence contradicts a dictum (which is ascribed to S. Weinberg) saying that a Poincaré invariant unitary S-matrix with these properties is characteristic for a (local) QFT.”

Relativity and Quantum Theory

- Inertial coordinate systems are coordinate systems where free particles move with constant velocity.
- Experiments on isolated systems cannot make an absolute determination of inertial coordinate system.

Relativity

- Because solutions of the Schrödinger equation are not observable in quantum theory, relativity does not require that solutions of the Schrödinger equation transform covariantly.
- This is an important difference between the classical and quantum formulation of relativistic invariance.

Relativistic Invariance

- Galilean Relativity: X and X' inertial $\Rightarrow$

$$|\Delta\vec{x}_{ij}| = |\Delta\vec{x}_{ij}{}'|$$

$$\Delta t_{ij} = \Delta t'_{ij}$$

- Special Relativity: X and X' inertial $\Rightarrow$

$$|\Delta\vec{x}_{ij}|^2 - c^2|\Delta t_{ij}|^2 = |\Delta\vec{x}'_{ij}|^2 - c^2|\Delta t'_{ij}|^2$$

Relativity

- The Michelson-Morley experiment verified that special relativity gives the observed relation between inertial coordinate systems.
- The most general transformation relating two inertial coordinate systems is a Poincaré transformation:

$$x^{\mu} \longrightarrow x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu}$$

$$g^{\mu\nu} = \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} g^{\alpha\beta}$$

Quantum Theory

- States are represented by unit vectors (rays), $|\psi\rangle$, in a complex vector space

$$\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$$

- The predictions of a quantum theory are the probabilities

$$P_{\phi\psi} = |\langle\phi|\psi\rangle|^2$$

Relativity in Quantum Theory

$$X \to X'$$

$$|\phi\rangle, |\psi\rangle \to |\phi'\rangle, |\psi'\rangle$$

$$\Downarrow$$

$$\boxed{|\langle\phi|\psi\rangle|^2 = |\langle\phi'|\psi'\rangle|^2}$$

- Must hold for all $|\psi\rangle$, $|\phi\rangle$ and all inertial coordinate systems X and X'

Wigner's Theorem

$$|\langle \phi | \psi \rangle|^2 = |\langle \phi' | \psi' \rangle|^2$$

$$\Downarrow$$

$$|\phi'\rangle = U|\phi\rangle \qquad |\psi'\rangle = U|\psi\rangle \qquad \langle \psi' | \phi' \rangle = \langle \psi | \phi \rangle$$

or

$$|\phi'\rangle = A|\phi\rangle \qquad |\psi'\rangle = A|\psi\rangle \qquad \langle \psi' | \phi' \rangle = \langle \psi | \phi \rangle^*$$

Wigner's Theorem

- For rotations, translations, and rotationless Lorentz transformations:

$$U = U^{1/2}U^{1/2} \qquad A = A^{1/2}A^{1/2} = U$$

- The correspondence between states must be unitary!

$$\Downarrow$$

$$|\psi'\rangle = U|\psi\rangle$$

Wigner’s Theorem

$$\underbrace{X \underbrace{\;\longrightarrow\;}_{(\Lambda,a)} X' \underbrace{\;\longrightarrow\;}_{(\Lambda',a')} X''}_{(\Lambda'',a'')}$$

$$\Downarrow$$

$$U(\Lambda', a')U(\Lambda, a) = U(\Lambda'\Lambda, \Lambda' a + a')$$

- $U(\Lambda, a)$ is a unitary representation of the Poincaré group

Elements of RQM

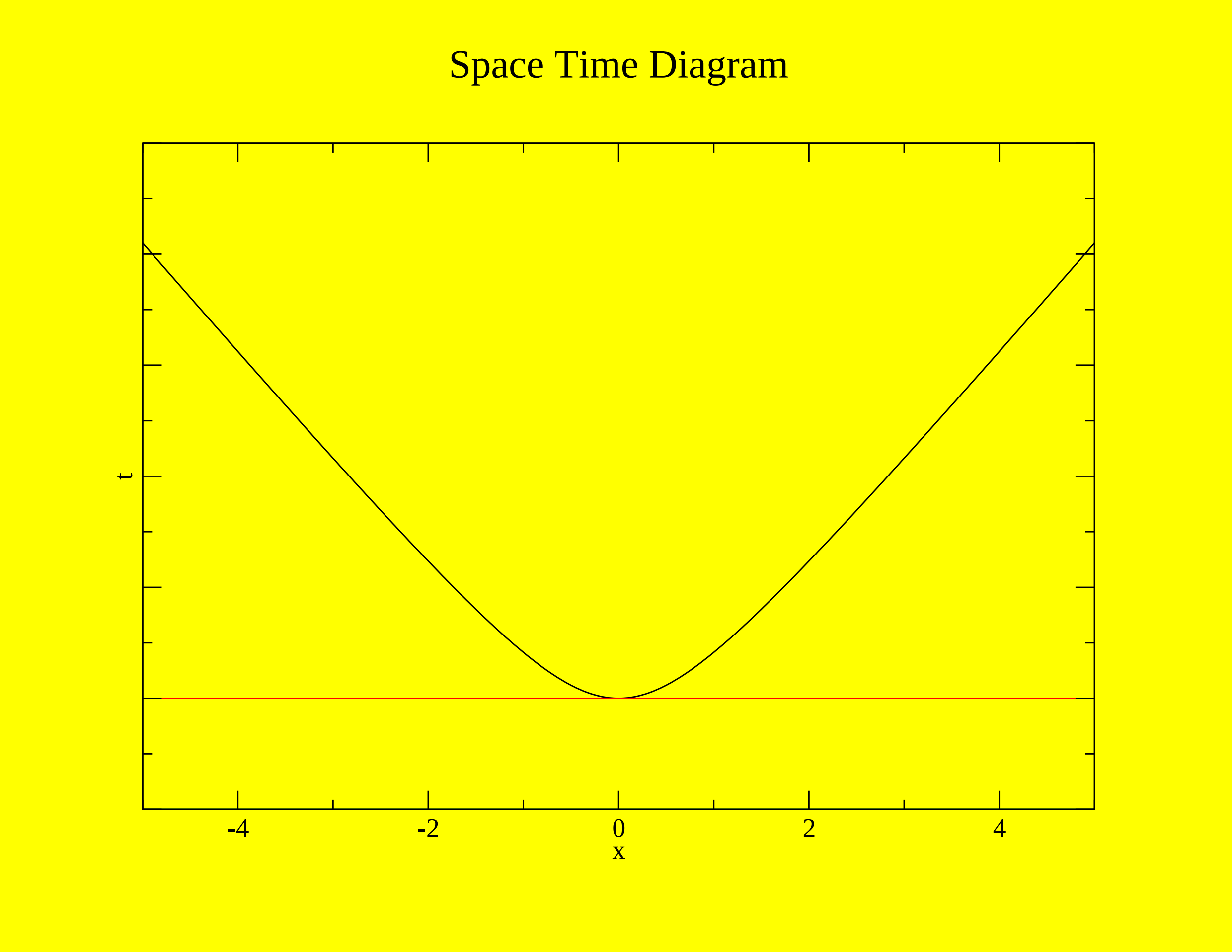
- Model Hilbert space: $\mathcal{H}$
- Unitary representation of Poincaré group: $U(\Lambda, a) : \mathcal{H} \to \mathcal{H}$
- Spectral condition: $H \geq 0 \qquad U(I,t) = e^{-iHt}$
- Cluster properties:

$$\lim_{|\vec{d}_1 - \vec{d}_2| \to \infty} \| [U(\Lambda,a) - U_1(\Lambda,a) \otimes U_2(\Lambda,a)] U_1(I,\vec{d}_1) \otimes U_2(I,\vec{d}_2) |\psi\rangle \| = 0$$

Possible approaches

- Covariant wave functions → quasi-Wightman functions.
- Covariant constraint dynamics → quasi-Wightman functions based on first class constraints.
- Euclidean relativistic quantum theory - based on reflection positive quasi-Schwinger functions.
- Directly interacting particles (this talk).

Difficulties



Difficulties

$$\Lambda = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix}$$

$$\vec{a} = (0,0,0,z) \quad \vec{b} = (0,0,0,-\gamma z) \quad t := (\gamma\beta z, \vec{0})$$

$$(\Lambda^{-1}, \vec{b})(\Lambda, \vec{a}) = (I, (t, \vec{0}))$$

Difficulties

$$U(\Lambda, a) = e^{-i\sum \lambda_i G_i}$$

$$G_i = G_i^\dagger \qquad G_i \in \{H, \vec{P}, \vec{J}, \vec{K}\}$$

$$[G_i, G_j] = i g_{ijk} G_k$$

$$\Downarrow$$

$$[P_i, K_j] = i\delta_{ij} H$$

Dynamics

- Galilean Invariant Dynamics $\{H, \vec{P}, \vec{J}, \vec{K}, M\}$

$$[P_i, K_j] = i\delta_{ij} M$$

- Poincaré Invariant Dynamics $\{H, \vec{P}, \vec{J}, \vec{K}\}$

$$[P_i, K_j] = i\delta_{ij}(H_0 + V)$$

- Puts non-linear constraints on $\{H, \vec{P}, \vec{J}, \vec{K}\}$ for consistent initial value problem!

The Currie-Jordan-Sudarshan Theorem

$$\{X_i, P_j\} = \delta_{ij}$$

$$\{X_i, J_j\} = \epsilon_{ijk} X_k$$

$$\{X_i, K_j\} = X_j \{X_i, H\}$$ world-line condition

- Poisson brackets of $\{H, \vec{P}, \vec{J}, \vec{K}\}$ satisfy Lie algebra of Poincaré group.

The Currie-Jordan-Sudarshan Theorem

$$\Downarrow$$

- Can only be satisfied for free particles!
- The CJS theorem suggests that it might be difficult to formulate relativistic models for systems of interacting particles.
- The out in relativistic quantum mechanics is that there is no position operator!

Position

- Let $|\vec{0}, 0\rangle$ be a vector corresponding to a particle at $\vec{x} = \vec{0}$ at time $t = 0$

$$\langle \vec{p} | \vec{0}; 0\rangle = \langle \vec{p} | U^{\dagger}(\Lambda(\vec{p}), 0) | \vec{0}; 0\rangle$$

$$= \frac{\sqrt{m}}{(\vec{p}^{\,2} + m^2)^{1/4}} \langle \vec{p} = 0 | \vec{0}; 0\rangle$$

$$= \frac{N}{(\vec{p}^{\,2} + m^2)^{1/4}}$$

$$\langle \vec{p} | \vec{x}; t = 0\rangle = \langle \vec{p} | U(I, \vec{x}\,) | \vec{0}; 0\rangle$$

$$\Downarrow$$

Position

$$\langle \vec{p}\,|\vec{x}\,\rangle = N \frac{e^{-i\vec{p}\cdot\vec{x}}}{(\vec{p}^{\,2}+m^2)^{1/4}}$$

$$\langle \vec{0}|\vec{x}\rangle = N^2 \int \frac{d^3p}{(\vec{p}^{\,2}+m^2)^{1/2}} e^{-i\vec{p}\cdot\vec{x}} =$$

$$\text{const} \times D_+(0,|\vec{x}|) \sim \left(\frac{mc|\vec{x}|}{\hbar}\right) e^{\frac{-mc|\vec{x}|}{\hbar}} \neq 0$$

Cluster Properties

$$\vec{K} = \vec{K}_{(12)(3)} + \vec{K}_{(23)(1)} + \vec{K}_{(31)(2)} - 2\vec{K}_0$$

$$[K^1, K^2] = -iJ^3 = -iJ_0^3$$

$$[K^1, K^2] = \cdots [K^1_{(12)(3)}, K^2_{(23)(1)}] + \cdots$$

- Commutator of interacting generators create many-body interactions that need to be canceled.
- Result is that either some cluster limits do not exist or quantities that should survive cluster limits actually vanish.

Cluster Properties

$$\lim_{\lambda\to\infty} e^{i\vec{p}_3\cdot\lambda\vec{a}} V_{12} e^{-i\vec{p}_3\cdot\lambda\vec{a}} = ?$$

$$\vec{p}_3 = \vec{q}_3 + \vec{P}F(m_{012},\vec{P},\vec{q}) =$$

$$\lim_{\lambda\to\infty} e^{\vec{P}\cdot\lambda\vec{a}F(m_{012},\vec{P},\vec{q})} V_{12} e^{-i\vec{P}\cdot\lambda\vec{a}F(m_{012},\vec{P},\vec{q}_3)}$$

$$[V_{12},\vec{q}_3] = [V_{12},\vec{P}] = 0; \qquad [V_{12}, m_{120}] \neq 0$$

$$\Downarrow$$

$$\lim_{\lambda\to\infty} e^{i\vec{p}_3\cdot\lambda\vec{a}} V_{12} e^{-i\vec{p}_3\cdot\lambda\vec{a}} = 0$$

Model Hilbert space

- Determine a complete measurement
- Defines a complete set of commuting observables
- The model Hilbert space is the space of square summable functions of the eigenvalues of the complete set of commuting observables.

$$\Downarrow$$

- Adequate to describe results of any experiment

One-Particle Relativistic Quantum Mechanics

$$U_1(\Lambda, a) \Rightarrow \{H, \vec{P}, \vec{J}, \vec{K}\}$$

$$\Downarrow$$

$$\{M, j, \eta_1, \cdots, \eta_4, \Delta\eta_1, \cdots, \Delta\eta_4\} \qquad [\eta_i, \eta_j]_- = 0$$

$$\Downarrow$$

$$\mathcal{H}_1 : \langle\psi|\phi\rangle = \sum \int \psi^*(\eta)\phi(\eta)d\eta$$

$$\Downarrow$$

$$U_1(\Lambda, a)|(m, j)\eta\rangle = \sum \int |(m, j)\eta'\rangle D^{m,j}_{\eta'\eta}(\Lambda, a)$$

Two-particle Hilbert space

$$\mathcal{H} := \mathcal{H}_1 \otimes \mathcal{H}_1 \qquad U_0(\Lambda, a) := U_1(\Lambda, a) \otimes U_1(\Lambda, a)$$

$$\Downarrow$$

$$G_0^i = G_1^i \otimes I_2 + I_1 \otimes G_2^i$$

$$\Downarrow$$

$$\{H_0, \vec{P}_0, \vec{J}_0, \vec{K}_0\}$$

$$\Downarrow$$

Two-particle Hilbert space

$$\{M, j, \eta_1, \cdots, \eta_4, \Delta\eta_1, \cdots, \Delta\eta_4\}$$

$$\boxed{U_1(\Lambda, a) \otimes U_1(\Lambda, a) = \int_{\oplus} U_{0,mj}(\Lambda, a)}$$

$$\langle (m_1, j_1)\eta_1; (m_2, j_2)\eta_2 | (m, j)\eta, \{d\}\rangle$$

$$U_0(\Lambda, a)|(m, j)\eta, \{d\}\rangle = \sum \int |(m, j)\eta', \{d\}\rangle D^{m,j}_{\eta'\eta}(\Lambda, a$$

Two particle Hilbert space

- The form of the Clebsch-Gordan coefficients depend on choice of vector labels η. There are an infinite number of choices.
- The two-body m has continuous spectrum; it is often replaced by

$$k \quad \Leftrightarrow \quad m = \sqrt{k^2 + m_1^2} + \sqrt{k^2 + m_2^2}$$

- Poincaré Clebsch-Gordan coefficients have multiplicity quantum numbers $\{d\}$.

Two-Particle Dynamics

$$\{M_0, j_0, \eta_{01}, \cdots, \eta_{04}, \Delta\eta_{01}, \cdots, \Delta\eta_{04}\}$$

$$\Downarrow$$

$$M = M_0 + V; \qquad 0 = [V, j_0] = [V, \eta_{0i}] = [V, \Delta\eta_{0i}]$$

- Find simultaneous eigenstates of $\{M, j_0, \eta_{0i}\}$
- Can be solved in basis of eigenstates of $\{M_0, j_0, \eta_{0i}\}$

$$\Downarrow$$

Two-Particle Dynamics

$$\langle (\underbrace{m_0}_{k}, j)\eta, \{d\} | (m, j')\eta' \rangle = \delta(\eta - \eta')\delta_{jj'}\phi_{m,j}(\underbrace{m_0}_{k}, \{d\})$$

$$\Downarrow$$

$$\boxed{U(\Lambda, a)|(m, j)\eta\rangle = \sum \int |(m, j)\eta'\rangle D^{m,j}_{\eta'\eta}(\Lambda, a)}$$

2+1-Particle Dynamics

$$U_{12}(\Lambda, a) \otimes U_3(\Lambda, a)$$

$$U_0(\Lambda, a) := U_{012}(\Lambda, a) \otimes U_3(\Lambda, a)$$

$$\Downarrow$$

$$|(m_{0123}, j_{0123})\eta, \{k_{12}, \{d_{12}\} \cdots\}\rangle$$

$$\bar{V} : [\bar{V}, k_{12}] \neq 0, \quad [\bar{V}, \{d_{12}\}] \neq 0, \quad [\bar{V}, \cdots] = 0$$

$$\Downarrow$$

2+1-Particle Dynamics

$$\bar{m}_{12} = m_{012} + \bar{V}$$

$$\bar{m}_{(12)(3)} = \sqrt{k_{0(12)(3)}^2 + \bar{m}_{12}^2} + \sqrt{k_{0(12)(3)}^2 + m_3^2}$$

$$\Downarrow$$

$$U_{12}(\Lambda, a) \otimes U_3(\Lambda, a) \qquad \bar{U}_{(12)(3)}(\Lambda, a)$$

2+1-Particle Dynamics

- $U_{12}(\Lambda, a) \otimes U_3(\Lambda, a)$ and $\bar{U}_{(12)(3)}(\Lambda, a)$ give identical scattering matrix elements.
- $U_{12}(\Lambda, a) \otimes U_3(\Lambda, a)$ clusters; combining different interactions destroys the Poincaré commutation relations.
- $\bar{U}_{(12)(3)}(\Lambda, a)$ violates cluster properties; Poincaré invariant addition of interactions for different interacting pairs possible in this representation.

Scattering Equivalences

$$A^\dagger = A^{-1} \qquad \lim_{t\to\pm\infty} \|(I - A)U_0(I,t)|\psi\rangle\| = 0$$

$$S(H, H_0) = \Omega_+^\dagger(H, H_0)\Omega_-(H, H_0)$$

$$\Downarrow$$

$$S(H, H_0) = S(H', H_0) \Leftrightarrow H' = AHA^\dagger$$

Scattering Equivalences

- Scattering equivalences A are unitary elements of a $*$ algebra of asymptotic constants.
- The $*$ algebra provides a functional calculus to construct functions of non-commuting scattering equivalences.
- Operators in this $*$ algebra relate $U_{12}(\Lambda, a) \otimes U_3(\Lambda, a)$ and $\bar{U}_{(12)(3)}(\Lambda, a)$

Cluster properties ⇔ irreps

$$\begin{tikzcd}
{|(12) \otimes (3)\rangle} \arrow[r, "\langle AB|C\rangle_0"] \arrow[d, "V_{(12)(3)}"'] & {|((12)(3))\rangle} \arrow[d, "\bar{V}_{((12)(3))}"] \\
{|(12)_I \otimes (3)\rangle} \arrow[r, "\langle AB|C\rangle_I"'] & {|((12)_I(3))\rangle \underbrace{\sim}_{A_{(12)(3)}} \overline{|((12)(3))_I\rangle}}
\end{tikzcd}$$

- $A_{(12)(3)}$ scattering equivalence

Three Particles

$$M = M_{(12)(3)} + M_{(23)(1)} + M_{(31)(2)} - 2M_0 + V_{(123)} =$$

$$\boxed{A^\dagger [\sum_{(ij)(k)} \underbrace{A_{(1j)(k)} M_{(ij)(k)} A^\dagger_{(ij)(k)}}_{\bar{M}_{(ij)(k)}} - 2M_0 + \bar{V}_{(123)}] A}$$

$$\vdots$$

$$H_{(ij)(k)} := H_{ij} \otimes I_k + I_{ij} \otimes H_k$$

$$A \rightarrow A_{(ij)(k)} \qquad A = \exp(\sum \ln(A_{(ij)(k)}))$$

Three Particles

- Result satisfies cluster properties and Poincaré Lie algebra.
- A is also a scattering equivalence.
- A and $A_{(ij)(k)}$ generate the many-body interactions needed to preserve commutation relations and cluster properties.
- Simplest non-trivial problem is electron scattering off of three-nucleon system.

Beyond-three

- Construction on previous slide is the first step in an inductive construction for N particles.
- The input is Poincaré Clebsch-Gordan coefficients, irreducible representations of Poincaré group, Poincaré Racah coefficients ($N \geq 3$), Poincaré Wigner-Eckart theorem (tensor and spinor operators), and Scattering equivalences generated by adding interacting and reducing representations in different orders.

Beyond-three

- The construction has been generalized to treat N-particle systems and systems with limited particle production.
- The requirement of cluster properties adds new dynamical non-linear constraints for more than two particles. These require many body interactions.

Beyond-three

- $U(\Lambda, a)$ implies that spinor, tensor, and four vector operators are interaction dependent. These can be constructed using the Poincaré Wigner-Eckart theorem
- Applications to few-body systems exist.

Outlook

- True production is an open problem.
- Need few degree of freedom problem directly related to both experiment and many-body problem.
- Use physical particles as degrees of freedom?
- Use center of momentum energy to control number of degrees of freedom?