Puzzles on the light front

W. N. Polyzou

The University of Iowa

Research supported by the U.S. Department of Energy, Office of Science, Grant #DE-SC0016457

• A light front is a three dimensional plane that is tangent to a light cone. It is the set of points $x^+ = t + \hat{\mathbf{n}} \cdot \mathbf{x} = 0$.

• There is a seven parameter subgroup of the Poincaré group that leaves this hyperplane unchanged - three translations in the plane, three boosts, and rotations about the $\hat{\mathbf{n}}$ axis.

• There are three additional independent Poincaré generators that do not leave the hyperplane unchanged, translations in the $x^+ = t + \hat{\mathbf{n}} \cdot \mathbf{x}$ direction and rotations about the two axes $\perp$ to $\hat{\mathbf{n}}$.

• Dirac proposed several methods for constructing relativistic Hamiltonian models that start with a non-interacting representation of the Poincaré Lie algebra. He identified sub algebras that do not involve the Hamiltonian and adds interactions to the remaining generators.

• In light front dynamics the non-interacting sub-algebra is given by the generators of transformations that leave the light front invariant.

Physics

- Useful for constructing electroweak or gravitational current matrix elements

$$\langle (m',s')\mathbf{p}',\mu' | J^{\mu}(0) | (m,s)\mathbf{p},\mu\rangle =$$

$$\langle (m',s')\mathbf{p},\mu'' | U^{\dagger}(\Lambda) J^{\mu}(0) | (m,s)\mathbf{p},\mu\rangle \qquad p' = \Lambda p$$

- Light front preserving boosts are kinematic.

$$\langle (m,s)\mathbf{p},\mu | \overrightarrow{U}(\Lambda) | \psi \rangle = \langle (m,s)\mathbf{p},\mu | \overleftarrow{U}_0(\Lambda) | \psi \rangle$$

- Light front boosts form a subgroup of the Poincaré group so the spins do not Wigner rotate.

- The vacuum does not change - the theory can be formulated on the free field Fock space.

- It is a Hamiltonian formulation of quantum field theory so non-perturbative problems are reduced to linear algebra (plus renormalization).

Light Front Quantization

- Independent of Dirac it was known that vacuum diagrams were suppressed by Lorentz transforming to a reference frame with large momentum. This proved to be an important simplification.

- Light front formulations of field theory were formulated and found to be "equivalent" to the infinite momentum limit of conventional formulations of field theory.

- Quantization: Use Noether's theorem to construct conserved currents from Poincaré invariance (energy momentum and angular momentum tensors). Integrate over the light front to construct charges = Poincaré generators. The generators that leave the light front invariant have no interactions. The fields restricted to the light front are irreducible and all of the generators can be expressed in terms of this operators in this irreducible algebra.

- Because of the irreducibility, initial data can be defined in terms of the light-front algebra and can be evolved off of the light front using dynamical generators.

- There is a spectral condition $P^+ = H - \mathbf{P} \cdot \hat{\mathbf{n}} \geq 0$ that suggests that interactions cannot change the vacuum.

- Unitary representations of the Poincaré Group

$$U(\Lambda_2, a_2)U(\Lambda_1, a_1) = U(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2)$$

implies that the infinitesimal generators satisfy the commutation relations:

$$[P^\mu, P^\nu] = 0, \qquad [J^i, P^j] = i\epsilon^{ijk} P^k, \qquad [J^i, J^j] = i\epsilon^{ijk} J^k,$$

$$[J^i, K^j] = i\epsilon^{ijk} K^k, \qquad [K^i, K^j] = -i\epsilon^{ijk} J^k$$

$$[K^i, P^i] = i\delta^{ij} H \qquad [K^i, H] = iP^i.$$

Light front generators are linear combinations of these operators. The relativistic analog of diagonalizing the Hamiltonian is to decompose $U(\Lambda, a)$ into a direct integral of irreducible representations

$$U(\Lambda, a) = \int_\oplus U_{ms}(\Lambda, a)$$

This is equivalent to simultaneously diagonalizing the mass and spin Casimir operators of the Lie algebra

$$M^2 = (P^0)^2 - \mathbf{P}^2 \qquad \text{and} \qquad \mathbf{S}^2 = W^2/M^2$$

where W^μ is the Pauli Lubanski vector

$$W^\mu = (\mathbf{P} \cdot \mathbf{J}, H\mathbf{J} + \mathbf{P} \times \mathbf{K}).$$

- Dirac's forms of dynamics:
- Dirac's instant-form dynamics (3 dimensional Euclidean group kinematic)

$$\mathbf{P} = \mathbf{P}_0 \qquad \mathbf{J} = \mathbf{J}_0.$$

$$[P^i, P^j] = 0, \qquad [J^i, P^j] = i\epsilon^{ijk} P^k, \qquad [J^i, J^j] = i\epsilon^{ijk} J^k,$$

- Dirac's point-form dynamics (Lorentz algebra kinematic)

$$\mathbf{J} = \mathbf{J}_0 \qquad \mathbf{K} = \mathbf{K}_0.$$

$$[J^i, K^j] = i\epsilon^{ijk} K^k, \qquad [K^i, K^j] = -i\epsilon^{ijk} J^k, \qquad [J^i, J^j] = i\epsilon^{ijk} J^k.$$

- Dirac's light-front dynamics (light front algebra kinematic)

$$P^1 = P_0^1, P^2 = P_0^2, P^+ = P_0^+ = P^0 + P^3, J^3 = J_0^3, K^3 = K_0^3, \mathbf{E}_\perp = \mathbf{E}_{0\perp} = \mathbf{K}_\perp - \hat{\mathbf{z}} \times \mathbf{J}$$

kinematic, while

$$P^- = P^0 - P^3 \neq P_0^-; \qquad \text{and} \qquad \mathbf{F}_\perp := \mathbf{K}_\perp + \hat{\mathbf{z}} \times \mathbf{J} \neq \mathbf{F}_{\perp 0} \qquad \text{or} \qquad \mathbf{J}_\perp = \hat{\mathbf{z}} \times \mathbf{J}$$

dynamical.

Puzzles

- The problem of inequivalent representations of the canonical commutation relations.

- The problem of the initial value problem.

- The vacuum problem.

- The problem of 0 modes.

- The problem of rotational covariance.

- The problem spontaneously broken symmetries.

• The problem of inequivalent representations of the canonical commutation relations.

Consider a pair of quantum harmonic oscillators with different frequencies with creation and annihilation operators related to coordinates and momenta by:

$$q = \frac{1}{\sqrt{2\omega_i}}\left(a_i + a_i^\dagger\right) \qquad p = -i\sqrt{\frac{\omega_i}{2}}\left(a_i - a_i^\dagger\right)$$

where ω_i is the angular frequency of the $i-th$ oscillator.

The creation and annihilation operator of the oscillators are related by a canonical transformation that preserves $[p_m, q_n] = i\delta_{mn}$:

$$a_2 = \cosh(\eta)a_1 + \sinh(\eta)a_1^\dagger$$

$$\cosh(\eta) := \frac{1}{2}\left(\sqrt{\frac{\omega_2}{\omega_1}} + \sqrt{\frac{\omega_1}{\omega_2}}\right) \qquad \sinh(\eta) := \frac{1}{2}\left(\sqrt{\frac{\omega_2}{\omega_1}} - \sqrt{\frac{\omega_1}{\omega_2}}\right).$$

This can be implemented by the following unitary transformation

$$e^{iG} \qquad \text{where} \qquad G = G^\dagger = -\frac{i}{2}\eta(a_1a_1 - a_1^\dagger a_1^\dagger).$$

The ground state vectors of the two oscillators are related by

$$|0\rangle_2 = e^{iG}|0\rangle_1.$$

•Free canonical fields behave like infinite collections of harmonics oscillators,

$$q_i \to \phi(\mathbf{x}, t), \qquad p_i \to \pi(\mathbf{x}, t), \qquad i \to \mathbf{x}$$

$$q_i \to \phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{\sqrt{2\omega_{m_i}(\mathbf{p})}} \left(e^{ip\cdot x} a_i(\mathbf{p}) + e^{-ip\cdot x} a_i^\dagger(\mathbf{p}) \right)$$

$$p_i \to \pi(x) = -\frac{i}{(2\pi)^{3/2}} \int d\mathbf{p} \sqrt{\frac{\omega_{m_i}(\mathbf{p})}{2}} \left(e^{ip\cdot x} a_i(\mathbf{p}) - e^{-ip\cdot x} a_i^\dagger(\mathbf{p}) \right)$$

where $\omega_m(\mathbf{p}) := \sqrt{m^2 + \mathbf{p}^2}$ is the energy of the particle

$$[p_i, q_j] = i\delta_{ij} \qquad \to \qquad [\pi(\mathbf{x}, t)\phi(\mathbf{y}, t)] = i\delta(\mathbf{x} - \mathbf{y})$$

$$\{|f_n\rangle\}_{n=1}^{\infty} \qquad \langle f_n|f_m\rangle = \delta_{mn} \qquad \text{orthonormal basis}$$

$$p_n := \int d\mathbf{x} f_n(\mathbf{x})\pi(\mathbf{x}, t) \qquad q_n := \int d\mathbf{x} f_n(\mathbf{x})\phi(\mathbf{x}, t) \qquad [p_n, q_m] = i\delta_{mn}$$

• Free fields with different masses are related by a canonical transformation

$$a_2(\mathbf{p}) = \cosh(\eta(\mathbf{p}))a_1(\mathbf{p}) + \sinh(\eta(\mathbf{p}))a_1^\dagger(\mathbf{p})$$

$$\cosh(\eta(\mathbf{p})) := \frac{1}{2}\left(\sqrt{\frac{\omega_{m_2}(\mathbf{p})}{\omega_{m_1}(\mathbf{p})}} + \sqrt{\frac{\omega_{m_1}(\mathbf{p})}{\omega_{m_2}(\mathbf{p}}}\right)$$

$$G = -\frac{i}{2}\int \eta(\mathbf{p})(a_1(\mathbf{p})a_1(\mathbf{p}) - a_1^\dagger(\mathbf{p})a_1^\dagger(\mathbf{p}))d\mathbf{p}.$$

and the vacuum vectors in the two theories would be related by

$$|0\rangle_2 = e^{iG}|0\rangle_1,$$

however a straightforward calculation gives

$$\| G|0\rangle_1\|^2 = \frac{1}{4}\int \eta(\mathbf{p})^2 d\mathbf{p}\delta(0) = \infty.$$

• Free fields with different masses live in Hilbert spaces that involve inequivalent representations of canonical commutation relations.

• A free field can be transformed to a light front field by changing variables from three momentum to light front momenta, $\tilde{\mathbf{p}} := (p^+, \mathbf{p}_\perp)$.

$$|\frac{\partial(\tilde{\mathbf{p}})}{\partial(\mathbf{p})}| = \frac{p^+}{\omega_m(\mathbf{p})} \qquad a(\tilde{\mathbf{p}}) := a(\mathbf{p})\sqrt{\frac{\omega_m(\mathbf{p})}{p^+}}$$

$$[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = \delta(\mathbf{p} - \mathbf{p}')$$

gives

$$[a(\tilde{\mathbf{p}}), a^\dagger(\tilde{\mathbf{p}}')] = \delta(\tilde{\mathbf{p}} - \tilde{\mathbf{p}}') = \delta(\mathbf{p}_\perp - \mathbf{p}'_\perp)\delta(p^+ - p^{+\prime})$$

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}_\perp dp^+ \theta(p^+)}{\sqrt{2p^+}} \left(e^{ip\cdot x} a(\tilde{\mathbf{p}}) + e^{-ip\cdot x} a^\dagger(\tilde{\mathbf{p}}) \right)$$

$$p^- = \frac{m^2 + \mathbf{p}_\perp^2}{p^+} \qquad p \cdot x = -\frac{1}{2}(p^+ x^- + p^- x^+) + \mathbf{p}_\perp \cdot \mathbf{x}_\perp .$$

• Note the mass dependence disappears when the field is restricted to the light front.

• Irreducibility - extracting creation and annihilation operators

Canonical case

$$a(\mathbf{p}) = \frac{1}{\sqrt{2\omega_{m_i}(\mathbf{p})}}(\omega_{m_i}(\mathbf{p})\hat{\phi}(\mathbf{p})_{x^0=0} + i\hat{\pi}(-\mathbf{p})_{x^0=0}),$$

$$a^\dagger(\mathbf{p}) = \frac{1}{\sqrt{2\omega_{m_i}(\mathbf{p})}}(\omega_{m_i}(\mathbf{p})\hat{\phi}(\mathbf{p})_{x^0=0} - i\hat{\pi}(-\mathbf{p})_{x^0=0}),$$

for free fields the mass is a dynamical quantity

Light front case

$$a(\tilde{\mathbf{p}}) = \sqrt{2p^+}\theta(p^+)\hat{\phi}(\tilde{\mathbf{p}})_{x^+=0} \qquad a^\dagger(\tilde{\mathbf{p}}) = \sqrt{2p^+}\theta(p^+)\hat{\phi}(-\tilde{\mathbf{p}})_{x^+=0}$$

In the light front case these are independent of mass. The transformed field has the same form independent of mass. The algebra of free fields of different mass restricted to the light front are unitarily equivalent.

• The puzzle is how to reconcile the unitary equivalence on the light front with the inequivalent representations for different masses in the canonical case.

- The problem of the initial value problem

The light front contains points that have a light like separation. This is along the line where the light front hyperplane intersects the light cone ($x^- = t - \hat{\mathbf{n}} \cdot \mathbf{x} = 0$). Since these points are causally connected in the light front, it cannot serve as an initial value surface.

Noether's theorem gives expressions for the Poincaré generators in terms of fields restricted to the light front. P^- is the generator of translations off of the light front. If there is a unitary representation of the Poincaré group P^- is a self-adjoint operator in the light front field algebra. The light front field algebra is irreducible so it can be used to represent initial data. This should result in a well-defined initial value problem.

- The problem of the initial value problem is how to reconcile these statements.

- The problem of the triviality of the vacuum:

$$P_0^+|0\rangle = 0 \qquad P_0^+ = \sum p_i^+ \qquad p_i^+ = \sqrt{\mathbf{p}_i^2 + m_i^2} + \hat{\mathbf{n}} \cdot \mathbf{p} \geq 0$$

Since P_0^+ commutes with both M and M_0 it commutes with the interaction $V := M - M_0$. It follows that

$$P_0^+ V|0\rangle = VP_0^+|0\rangle = 0.$$

Because of the spectral condition $P^+ \geq 0$ - the vacuum is the only normalizable eigenstate with $P^+ = 0$. This means that $|0\rangle$ and $V|0\rangle$ are both eigenstates of P_0^+ with eigenvalue 0. It follows that

$$\langle 0|V^\dagger V|0\rangle = \int |\langle p_0^+, d|V|0\rangle|^2 d\mu(p_0^+) dd$$

$$V|0\rangle = |0\rangle\langle 0|V|0\rangle$$

If the vacuum is the only discrete normalizable state of the theory that is invariant under translations on the light front then

$$0 = (P^- P^+ - \mathbf{P}_\perp^2)|0\rangle = M^2|0\rangle =$$

$$(M_0^2 + VM_0 + M_0V + V^2)|0\rangle = V^2|0\rangle = |0\rangle\langle 0|V|0\rangle^2 .$$

Then the constant must vanish.
The vacuum expectation value of a product of fields on the light front is

$$\langle 0|\phi(x)\phi(0)|0\rangle = \frac{1}{2(2\pi)^3}\int \frac{\theta(q^+)dq^+ d\mathbf{q}_\perp}{q^+} e^{-i\frac{q^+}{2}x^- + i\mathbf{q}_\perp\cdot\mathbf{x}_\perp} =$$

$$\frac{\delta(\mathbf{x}_\perp^2)}{4\pi}\int_0^\infty \frac{dq^+}{q^+} e^{-i\frac{q^+}{2}x^-}$$

which differs from the light-front limit of the full free field Wightman function is

$$\langle 0|\phi(x)\phi(0)|0\rangle \to -i\frac{\epsilon(x^-)\delta(\mathbf{x}_\perp^2)}{4\pi} - \frac{mK_1(m\sqrt{\mathbf{x}_\perp^2})}{4\pi^2\sqrt{\mathbf{x}_\perp^2}} .$$

• The problem of the triviality of the vacuum is that the vacuum in canonical theories is never the Fock vacuum. Interactions involve operators that create particles out of the vacuum.

• The problem of 0 modes.

Calculations of the dynamics using the light front Hamiltonian, P^-, that comes from Noether's theorem do not always agree with corresponding covariant calculations.

At the perturbative level these can be reconciled by including some $P^+ = 0$ (zero mode) contributions that are independent of the light front Hamiltonian.

• The problem of 0 modes is how to define a consistent light-front dynamics that agrees with the results of canonical field theory without having to appeal to those results.

• The problem of rotational covariance

The operators $\{P^+, E^1, E^2, K^3, J^3, P^1, P^2, P^-\}$ form a closed Lie algebra. It is not sufficient to generate the Poincaré Lie algebra.

If W is unitary and commutes with P^- and the kinematic generators and J^1, J^2 complete the full Poincaré Lie algebra then $J^{1\prime} = WJ^1W^\dagger$, $J^{2\prime} = WJ^2W^\dagger$ also complete the algebra. This means that J^1, J^2 are not uniquely fixed by P^-.

When calculations of eigenstates of P^- have degeneracies in magnetic quantum numbers - there is no unique way to assign spins.

• The problem of rotational covariance is how to formulate light front dynamics to get the correct rotational covariance of covariant field theory.

• The problem of spontaneous symmetry breaking

When the symmetry of the vacuum is spontaneous broken there is a 0 mass Goldstone boson in the mass spectrum. The charge operator Q that generates the symmetry does not leave the vacuum invariant. The condition for the existence of a 0 mass particle is

$$\langle 0|[Q_C, \phi(y)]|0\rangle \neq 0$$

Light front case spectral condition $P^+ \geq 0$ implies the charges always annihilates the vacuum

$$\langle 0|[Q_{LF}, \phi(y)]|0\rangle = 0$$

• The problem of spontaneous symmetry breaking is how to reconcile these two results.

• Resolutions. It should not be surprising that the resolution of these puzzles are related.

What is the vacuum?

• In algebraic quantum field theory the vacuum is a positive linear functional on the algebra.

• In light front field theory the vacuum satisfies $a(\tilde{p})|0\rangle = 0$.

• In a canonical field theory it has been shown that if the Hamiltonian is quadratic in the momentum field then the vacuum uniquely determines the rest of the Hamiltonian (Coester Haag Araki).

• Algebraically fields are operator valued distributions that transform covariantly and satisfy locality. Except for the types of fields, this is the same setting for all field theories. Vacuum expectation values of polynomials of these operators determine the elements (Wightman distributions) that separate different theories. In this sense the vacuum functional uniquely determines the dynamics from the abstract field algebra, just like in the canonical case.

• Vectors in the Hilbert space of the field theory involve integrating the operator valued distributions with Schwartz functions in four spacetime dimensions and applying functions of these operators to the vacuum (GNS construction).

• Smearing just over functions with support in a three dimensional hyperplane may give an incomplete characterization of vacuum expectation values of the same fields smeared over Schwartz functions in four spacetime variables.

• Free fields:

$$\phi(f) := \int \phi(x) f^*(x) d^4x = \int K_m(x, y^-, \mathbf{y}_\perp) \phi_{LF}(y^+ = 0, y^-, \mathbf{y}_\perp) f(x) d^4x d\tilde{\mathbf{y}}$$

$$\phi(x) = \int K_m(x, y^-, \mathbf{y}_\perp) \phi_{LF}(y^+ = 0, y^-, \mathbf{y}_\perp) d\tilde{\mathbf{y}}$$

$$\mathcal{K}_m(x, \tilde{\mathbf{y}}) := \int \frac{d\tilde{\mathbf{p}}}{(2\pi)^3} e^{-ix^+ \frac{\mathbf{p}_\perp^2 + m^2}{2p^+}} e^{i \frac{p^+ \cdot (y^- - x^-)}{2} - i\mathbf{p}_\perp \cdot (\mathbf{y}_\perp - \mathbf{x}_\perp)}$$

• $\mathcal{K}_m(x, \tilde{\mathbf{y}})$ defines a mapping from local fields to a sub algebra of fields on the light front.

For free fields all Wightman functions are products of two point functions.

The two point Wightman functions of the free field theory

$$\langle 0|\phi(f)\phi(g)|0\rangle =$$

$$\langle 0| \int K_m(x, \tilde{\mathbf{y}})\phi(\tilde{\mathbf{y}})f(x)d^4x d\tilde{\mathbf{y}} K_m(x', \tilde{\mathbf{y}}')\phi(\tilde{\mathbf{y}}')g(x')d^4x' d\tilde{\mathbf{y}}'|0\rangle =$$

$$\langle 0| \int F_{f,g}(\tilde{\mathbf{y}}, \tilde{\mathbf{y}}')\phi_{LF}(\tilde{\mathbf{y}})\phi_{LF}(\tilde{\mathbf{y}}')|0\rangle$$

can be expressed in terms of light front vacuum expectation values of a sub algebra of fields on the light front.

• For free fields restricted to the light-front vacuum, different vacuum functionals are unitarily equivalent on the full light front algebra, but not on the dynamical sub-algebras.

• Different kernels $\mathcal{K}_m$ generate mappings to different sub-algebras of the light front Fock algebera.

$$\phi(f) = \int \frac{dy^+ d\mathbf{y}_\perp}{2} f_m(\tilde{\mathbf{y}})\phi(\tilde{\mathbf{y}})_{|y^+=0} := \phi_{LF}(f_m)$$

This shows that operators in the Heisenberg algebra can be expressed as operators in the light front Fock algebra.

• The Fourier transform of $f_m(\tilde{\mathbf{y}})$ assuming the $f(x)$ is a real Schwartz function

$$\hat{f}_m(\tilde{\mathbf{p}})^* = \sqrt{2\pi}\hat{f}^*(\frac{\mathbf{p}_\perp^2 + m^2}{p^+}, \tilde{\mathbf{p}}).$$

vanishes exponentially as $p^+ \to 0$. This means that these function vanish at the non-causal points on the light front are eliminated (this resolves the initial value puzzle).

• The two point Wightman functions (the dynamics) can still be expressed as vacuum expectation values of elements in a sub algebra of the algebra of free fields on the light front.

• The smearing, which selects the sub algebra must be done first - there is nothing like the Wightmann distributions on the light front.

- This can be extended to interacting theories by combining:

$$\phi(x) = \sum \int d^{4n} y R(x, y_1 \cdots y_n) : \phi_{IN}(y_1) \cdots \phi_{IN}(y_N) :$$

- The IN fields behave like free fields with the physical masses:

$$\prod_i \phi_{INi}(y_i)|0\rangle = \prod_i (\int \mathcal{K}_m(x, \tilde{\mathbf{y}}) \phi_{LF}(\tilde{\mathbf{y}}) d\tilde{\mathbf{y}} |0\rangle$$

- Combining these formulas gives

$$\Downarrow$$

$$\langle 0|\phi_1(f_1) \cdots \phi_N(f_N)|0\rangle =$$

$$\int \sum_{LF} \prod_i d\tilde{\mathbf{y}}_i \langle 0| \sum_n O_n(f_1 \cdots f_N, \tilde{\mathbf{y}}_1 \cdots \tilde{\mathbf{y}}_n) \phi_{LF}(\tilde{\mathbf{y}}_1) \cdots \phi_{LF}(\tilde{\mathbf{y}}_N)|0\rangle$$

- This expresses the Wightman functions in terms of vacuum expectation values of elements of very a complicated sub-algebra of the light front field algebra.

- The problem of rotational covariance

Starting with one component of $\mathbf{J}_\perp$ and the kinematic generators it is possible to construct the full Poincaré Lie algebra:

$$P^- := P^+ - 2[J^2,[J^2,P^+]] \qquad \text{and} \qquad J^1 := -i[J^2,J^3].$$

The commutation relations imply constraints on $J^2 = J_0^2 + J_I^2$; J_I^2 commutes with all kinematic generators and must satisfy

$$[J_I^2,[J_0^2,P^1]] = 0$$

$$[J_I^2,[J_I^2,J^3]] + [J_0^2,[J_I^2,J^3]] + i[J_I^2,J_0^1] = 0.$$

Transverse rotations can be used to compute the field off of the light front in terms of quantities in the light front algebra:

$$U_y(\theta)\phi(\tilde{\mathbf{x}},0)U_y^\dagger(\theta) = \phi(\frac{1+\cos(\theta)}{2}x^-, -\sin(\theta)x_1, x_2, \frac{1-\cos(\theta)}{2}x^-) =$$

$$\sum_{n=0}^\infty \frac{(i\theta)^n}{n!} \underbrace{[J_2,[J_2,\cdots,[J_2,\phi(\tilde{\mathbf{x}},0)]]]}_{\text{N times}}.$$

This can be combined with kinematic translational invariance to get to any other point.

The structure of the dynamical operators in terms of light front creation and annihilation operators (free fields):

$$J^2 = \int d\tilde{\mathbf{p}}\theta(p^+)a^\dagger(\tilde{\mathbf{p}}) \left(-\frac{1}{2}p^+(i\partial_{p_1}) + \frac{1}{4}\{\frac{\mathbf{p}_\perp^2 + m^2}{p^+}, i\partial_{p_1}\} + \frac{1}{2}(2ip^1\partial_{p+}) \right) a(\tilde{\mathbf{p}})$$

$$P^- = \int d\tilde{\mathbf{p}}\theta(p^+)a^\dagger(\tilde{\mathbf{p}}) \frac{\mathbf{p}_\perp + m^2}{p^+} a(\tilde{\mathbf{p}})$$

$$J^1 = \int d\tilde{\mathbf{p}}\theta(p^+)a^\dagger(\tilde{\mathbf{p}}) \left(\frac{1}{2}p^+(i\partial_{p_2}) - \frac{1}{4}\{\frac{\mathbf{p}_\perp^2 + m^2}{p^+}, i\partial_{p_2}\} - \frac{1}{2}(2ip^2\partial_{p+}) \right) a(\tilde{\mathbf{p}})$$

- P^- alone is insufficient to define the full dynamics.

- There is more than one way to define transverse rotation operators to complete the Poincaré Lie algebra. Thus it is not sufficient to simply restore rotational covariance.

- One component of the transverse angular momentum defines the theory.

- Rotational covariance is equivalent to the requirement that the results are independent of inertial coordinate system. This mixes infrared and ultraviolet singularities. Constraining the renormalization of P^- so the results are independent of the choice of orientation of the light front can restore rotational covariance.

- A careful treatment of $P^+ = 0$ modes is needed to ensure rotational covariance.

- The problem of the initial value problem
- The mapping $\mathcal{K}_m$ intertwines covariant qft or lf qft:

$$i\frac{\partial}{\partial x^+}\hat{\mathcal{K}}_m(x,\tilde{\mathbf{p}}) = \hat{\mathcal{K}}_m(x,\tilde{\mathbf{p}})\frac{m^2+\mathbf{p}_\perp^2}{p^+}$$

$$\int \phi(x^+ - a^+, \tilde{\mathbf{x}}) f(x) d^4x =$$

$$\int d^4y d\tilde{\mathbf{p}} f(y) \sum_{n=0}^{\infty} \frac{(-a^+)^n}{n!} \frac{\partial^n}{\partial {y^+}^n} \mathcal{K}_m(y^+, \tilde{\mathbf{y}}, \tilde{\mathbf{p}}) \phi_{LF}(-\tilde{\mathbf{p}}) =$$

$$\sqrt{2\pi}\int d\tilde{\mathbf{p}} \sum_{n=0}^{\infty} \frac{(ia^+)^n}{n!} (\frac{\mathbf{p}^2 + m^s}{2p^+})^n \hat{f}(\frac{\mathbf{p}_\perp^2 + m^2}{p^+}, \tilde{\mathbf{p}}) \phi_{LF}(-\tilde{\mathbf{p}})$$

The series converges when the original Schwartz functions are restricted to the dense subspace of test functions with compact support:

$$\sum_{n=0}^{\infty} \frac{(a^+)}{n!} (\frac{\mathbf{p}_\perp^2 + m^2}{2p^+})^n \hat{f}(\frac{\mathbf{p}_\perp^2 + m^2}{p^+}, \tilde{\mathbf{p}})$$

The compactness provides the momentum and energy cutoff that eliminates problems due to the causally connected point on the lightfront. Here this makes sense on a dense subset of the sub-algebra.

- The problem of zero modes

- P^- and the kinematic generators do not completely define the theory.

- Non trivial vacua are formally due the part of the Hamiltonian that have all creation operators:

$$\int \frac{\theta(p^+)\delta(p^+)dp^+}{(p^+)^2\prod \xi_i} \prod d\mathbf{p}_{i\perp} d\xi_i \delta(\sum \mathbf{p}_{i\perp})\delta(\sum \xi_i - 1)\times$$
$$a^\dagger(\xi_1 p^+, \mathbf{p}_{\perp 1})a^\dagger(\xi_2 p^+, \mathbf{p}_{\perp 2})a^\dagger(\xi_3 p^+, \mathbf{p}_{\perp 3})a^\dagger(\xi_4 p^+, \mathbf{p}_{\perp 4}).$$

- These are very singular at $p^+ = 0$ which corresponds $\mathbf{p}\cdot\hat{\mathbf{z}} \to -\infty$. Renormalizing the $p^+ = 0$ singularities in the light front case is equivalent to renormalizing them ultraviolet singularities in the covariant case.

- The IR and UV singularities are related by rotational covariance.

• The problem spontaneously broken symmetry

• Current conservation:

$$\langle 0|[\int d\mathbf{x}\partial_\mu j^\mu(\mathbf{x},t),\phi(y)]|0\rangle = 0.$$

• The non-perturbative condition for the presence of a 0-mass Goldstone boson is:

$$\lim_{R\to\infty}\langle 0|[Q_R,\phi(y)]|0\rangle \neq 0$$

where

$$Q_R = \int d\mathbf{x}\chi_R(|\mathbf{x}|)j^0(\mathbf{x},t)$$

$$\langle 0|[Q_R,\phi(y)]|0\rangle := \langle 0|[\int d\mathbf{x}\chi_R(|\mathbf{x}|)j^0(\mathbf{x},t),\phi(y)]|0\rangle = \int d\mathbf{x}\langle 0|[j^0(\mathbf{x},t),\phi(y)]|0\rangle.$$

$$\langle 0|[\int d\mathbf{x}\chi_R(|\mathbf{x}|)\partial_\mu j^\mu(\mathbf{x},t),\phi(y)]|0\rangle = 0.$$

Condition makes sense by locality! Locality cannot be used on the light front.

Inserting a complete set of intermediate states gives

$$0 = \sum \int d\mathbf{x}\chi_R(|\mathbf{x}|)\left(\langle 0|\partial_\mu j^\mu(\mathbf{x},t)|p,n\rangle \frac{d\mathbf{p}}{2p_n^0}\langle p,n|\phi(y)|0\rangle\right.$$

$$\left. -\langle 0|\phi(y)|p,n\rangle \frac{d\mathbf{p}}{2p_n^0}\langle p,n|\partial_\mu j^\mu(\mathbf{x},t)|0\rangle\right) =$$

Using Lorentz invariance

$$\sigma(m_n)m_n^2 = \langle 0|\partial_\mu j^\mu(\mathbf{0},0)|p_r,n\rangle\langle p_{nr},n|\phi(0)|0\rangle$$

and

$$\sigma^*(m_n)m_n^2 = \langle 0|\phi(0)|p_{nr},n\rangle\langle p_{nr},n|\partial_\mu j^\mu(\mathbf{0},0)|0\rangle$$

$$0 = \int \sum(\sigma(m_n) - \sigma^*(m))m^2$$

$$\Downarrow$$

$$m_n^2 = 0 \qquad \text{or} \qquad Im(\sigma(m_n)) = 0$$

$$\langle 0|[\int d\mathbf{x}\chi_R(|\mathbf{x}|)j^0(\mathbf{x},t),\phi(y)]|0\rangle \neq 0.$$

$$\sum\int d\mathbf{x}\left(\langle 0|j^0(\mathbf{x},t)|p,n\rangle\frac{d\mathbf{p}}{2p_n^0}\langle p,n|\phi(y)|0\rangle\right.$$

$$\left.-\langle 0|\phi(y)|p,n\rangle\frac{d\mathbf{p}}{2p_n^0}\langle p,n|j^0(\mathbf{x},t)|0\rangle\right)\neq 0$$

$$\sigma(m_n)p_n^0 = \langle 0|j^0(\mathbf{0},0)|p_{nr},n\rangle\langle p_{nr},n|\phi(0)|0\rangle$$

and

$$\sigma^*(m_n)p_n^0 = \langle 0|\phi(0)|p_{nr},n\rangle\langle p_{nr},n|j^0(\mathbf{0},0)|0\rangle$$

$$\int\sum(\sigma(m_n)-\sigma^*(m_n))p_{nr}^0 \neq 0$$

Conclusions

- All vacuum functionals are unitarily equivalent on the light front, but not on the dynamical sub algebras.
- The light front field algebra is irreducible and as a result there are mappings from vacuum expectation values of products of Heisenberg fields to light front Fock vacuum expectation values of elements of a sub algebra of the light front Fock algebra.
- The dynamics is entirely due to the mappings; the inequivalent representations result from the different mappings.
- The mappings enforce the boundary conditions that eliminate the causally connected points. The different subalgebras are sub algebras of the Schlieder Seiler sub algebra with test functions $p^+ f(p)$.
- The light front Hamiltonian does not completely define the dynamics. Simply restoring rotational covariance is not sufficient. This requires the generators of transverse rotations or the field equations of a rotationally invariant renormalization.
- The triviality of the vacuum ignores infrared singularities in the dynamical generators. These must be renormalized in manner that preserves rotational covariance.