Surgery on the absolutely continuous spectrum

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Background:

- The many- body scattering problem is numerically intractable. In some experiments there is a high probability that most of the final states will be in a small number of dominant channels.

- To simplify the numerical problem is it desirable to have a first approximation where all of the scattered particles are in one of the dominant scattering channels.

- Scattering solutions have continuous energy eigenvalues. Removing open unimportant channels involves perturbing the absolutely continuous spectrum.

- Unlike the point spectrum (bound states) the absolutely continuous spectrum can be destroyed by arbitrarily small perturbations (Weyl Von-Neumann).

Remarks:

- Consider an integral equation of the form

$$X(z) = D(z) + K(z)X(z) \qquad K(z) = F(z) + \Delta(z) \qquad z = E + i\epsilon$$

$$X(z) = (I - F(z))^{-1} (D(z) + \Delta(z)X(z)) =$$

$$\sum_{n=0}^{\infty} \left((I - F(z))^{-1}\Delta(z)\right)^n (I - F(z))^{-1} D(z)$$

- For compact $K(z)$ we can choose $F(z)$ a finite dimensional matrix and $\Delta(z)$ small. Then $(I - F(z))^{-1}$ can only change the discrete spectrum. The continuous spectrum is in $D(z)$ (analytic Fredholm theorem).

- The problem is how to remove contributions of the unwanted channels from the continuous spectrum to $D(z)$.

- Scattering channels α .

$$|\Psi_{\alpha}^{(\pm)}\rangle = s - lim_{t\to\pm\infty} \int \sum_{\mu_1\cdots\mu_N} e^{iHt} e^{-iH_a t} \otimes_i |(E_i, j_i)\mathbf{p}_i, \mu_i\rangle \phi_i(\mathbf{p}_i, \mu_i) d\mathbf{p}_i$$

where

$$H_a = \sum_{i=1}^{n_a} H_{a_i} \qquad H_{a_i}|(E_i, j_i)\mathbf{p}_i, \mu_i\rangle = |(E_i, j_i)\mathbf{p}_i, \mu_i\rangle E_i$$

- Notation

$$\Phi_\alpha|\phi_{\alpha 0}\rangle := \sum_{\mu_1\cdots\mu_{n_a}} \int \otimes|(E_i, j_i)\mathbf{p}_i, \nu_i\rangle d\mathbf{p}_i \phi_i(\mathbf{p}_i, \nu_i) d\mathbf{p}_i.$$

$$\phi_{0\alpha}(\mathbf{p}_1, \nu_1, \cdots \mathbf{p}_{n_1}, \nu_{n_a}) = \prod_{j=1}^{n_a} \phi_j(\mathbf{p}_j, \nu_j).$$

$$|\Psi_{\alpha}^{(\pm)}\rangle = \lim_{t\to\pm\infty} e^{iHt} e^{-iH_a t} \Phi_\alpha|\phi_{0\alpha}\rangle = \Omega(a)^{(\pm)} \Phi_\alpha|\phi_{0\alpha}\rangle.$$

- Assumptions
  - Asymptotic completeness: $\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}^{(-)} = \mathcal{H}_B \oplus \mathcal{H}^{(+)}$ (=unitarity of the scattering matrix)
  - Short range interactions in H (Cook's condition sufficient):

$$\pm \int_0^{\pm\infty} \| \sum_{\mu_1 \cdots \mu_N} \underbrace{(H - H_a)}_{H^a} e^{-iH_a t} \otimes_i |(E_i, j_i)\mathbf{p}_i, \mu_i\rangle \phi_i(\mathbf{p}_i, \mu_i) d\mathbf{p}_i \| < \infty$$

  - Connected products of interactions, resolvents and bound state projection operators are compact on some Banach space (typical property of Faddeev-Yakubovskii formulations of scattering).
  - $\mathcal{A} =$ set of all scattering channels including the bound state channels.

- Scattering operator

$$S = I\delta_{\beta\alpha} - 2\pi i\delta(E_\beta - E_\alpha)T^{\beta\alpha}$$

- Transition operator

$$T^{\beta\alpha} = \Phi_\beta^\dagger H^b \Omega(a)^{(-)}\Phi_\alpha .$$

- Differential cross section

$$d\sigma = \frac{(2\pi)^4}{|s\mathbf{v}_r|}|\langle\beta, \mathbf{p}_1', \mu_1', \cdots, \mathbf{p}_{n_b}', \mu_{n_b}'\| T^{\beta\alpha}\|\mathbf{p}_1, \mu_1, \mathbf{p}_2, \mu_2\rangle|^2 \times$$

$$\delta(E_1 + E_2 - \sum_{j=1}^{n_b} E_j')\delta(\mathbf{p}_1 + \mathbf{p}_2 - \sum_{j=1}^{n_b} \mathbf{p}_j')\prod_{i=1}^{n_b} d\mathbf{p}_1'$$

- Cluster combinatorics
  - $\mathcal{P}$ is the set of partitions of an N particle system into non-empty disjoint subsystems.
  - n_a is the number of equivalence classes of a
  - a_i is the set of particles in the i^{th} equivalence class of a
  - n_{a_i} is the number of particles in the i^{th} equivalence class of a
  - $i \sim_a j$ means that particles i and j are in the same equivalence class of a
  - $0 := \{(1)(2)\cdots(N)\}$ is the unique N cluster partition (each particle in a different class)
  - $1 := \{(1\cdots N)\}$ is the unique 1 cluster partition (all particles in the same class).

$$\sum_{i=1}^{n_a} n_{a_i} = N$$

- (Birkhoff) Lattice structure
- partial ordering on partitions a

$$a \subseteq b \qquad i \sim_a j \rightarrow i \sim_b j$$

- Greatest lower bound and least upper bound with respect to $\subseteq$:

  $a \cup b$: $a \subseteq a \cup b$, $b \subseteq a \cup b$, and if $a \subseteq c$, $b \subseteq c$ then $a \cup b \subseteq c$

  $a \cap b$: $a \cap b \subseteq a$, $a \cap b \subseteq b$, and if $c \subseteq a$, $c \subseteq b$, then $a \cap b \subseteq c$

- Zeta function on incidence algebra ($\subseteq$):

$$\Delta_{a \supseteq b} = \begin{cases} 1 & a \supseteq b \\ 0 & a \not\supseteq b \end{cases}$$

- Möbius function on incidence algebra ($\subseteq$):

$$\Delta^{-1}_{a \supseteq b} = \begin{cases} (-)^{n_a} \prod_{i=1}^{n_a} (-)^{n_{b_i}} (n_{b_i} - 1)! & a \supseteq b \\ 0 & a \not\supseteq b \end{cases}$$

- Classification of operators (a=partition)

$$T_a(\mathbf{x}_1, \cdots, \mathbf{x}_{n_a}) := e^{-i\sum_{i=1}^{n_a} \mathbf{x}_i \cdot \mathbf{P}_{a_i}}. \qquad \mathbf{P}_{a_i} = \sum_{j \in a_i} \mathbf{p}_j$$

- Separates clusters of partition a

$$O = O_a + O^a$$

$$[O_a, T_a(\mathbf{x}_1, \cdots, \mathbf{x}_{n_a})] = 0 \qquad \lim_{|\mathbf{x}_i - \mathbf{x}_j| \to \infty} \| O^a T_a(\mathbf{x}_1 \cdots \mathbf{x}_{n_a}) |\psi\rangle \|. = 0.$$

- O_a= T_a-invariant part of O

$$O_a = \lim_{|\mathbf{x}_i - \mathbf{x}_j| \to \infty} T_a(\mathbf{x}_1 \cdots \mathbf{x}_{n_a}) O T_a^\dagger(\mathbf{x}_1 \cdots \mathbf{x}_{n_a})$$

For $a \subseteq b$ $T_b(\mathbf{x}_1, \cdots, \mathbf{x}_{n_b})$ is a subgroup of $T_a(\mathbf{x}_1, \cdots, \mathbf{x}_{n_a})$. It follows that

$$O_a = T_b(\mathbf{x}_1, \cdots, \mathbf{x}_{n_b}) O_a T_b^\dagger(\mathbf{x}_1, \cdots, \mathbf{x}_{n_b})$$

For $a \nsubseteq b$ then O_a has the following decomposition

$$O_a = (O_a)_b + (O_a)^b = O_{a\cap b} + O_a^b$$

$$\lim_{|\mathbf{x}_i - \mathbf{x}_j| \to \infty} \| T_b(\mathbf{x}_1, \cdots, \mathbf{x}_{n_b}) O_a^b T_b^\dagger(\mathbf{x}_1, \cdots, \mathbf{x}_{n_b}) |\psi\rangle \| = 0$$

$$O_{a\cap b} = \lim_{|\mathbf{x}_i - \mathbf{x}_j| \to \infty} T_b(\mathbf{x}_1, \cdots, \mathbf{x}_{n_b}) O_a T_b^\dagger(\mathbf{x}_1, \cdots, \mathbf{x}_{n_b})$$

- Definition: $[O]_a$ a-connected part of O.

$$[T_a, [O]_a] = 0 \qquad ([O]_a)_b = 0 \qquad \text{for} \qquad a \not\subset b.$$

It follows that

$$O_a = \sum_{a \subset b} [O]_b = \sum_b \Delta_{a \subset b} [O]_b \qquad [O]_a = \sum_b \Delta^{-1}_{a \subset b} O_b$$

$$[O]_1 = \sum_b \Delta^{-1}_{1 \subset b} O_b = \Delta^{-1}_{1 \subset 1} O_1 + \sum_{b \neq 1} \Delta^{-1}_{1 \subset b} O_b = O + \sum_{b \neq 1} \Delta^{-1}_{1 \subset b} O_b.$$

$$O = -\sum_{a \neq 1} \Delta^{-1}_{1 \subset b} O_a + [O]_1$$

$$\mathcal{C}_a := -\Delta^{-1}_{1 \subset a} = (-)^{n_a} (n_a - 1)!$$

$$\boxed{O = \sum_{a \neq 1} \mathcal{C}_a O_a + [O]_1}$$

$$(AB)_a = A_a B_a$$

$$AB - [AB]_1 = \sum_{a\neq 1} \mathcal{C}_a A_a B_a = (\sum_{a\neq 1} \mathcal{C}_a A_a + [A]_1)B - [AB]_1 =$$

$$(\sum_{a\neq 1} \mathcal{C}_a A_a + [A]_1)(B_a + B^a) - [AB]_1 =$$

$$\sum_{a\neq 1} \mathcal{C}_a A_a B_a + \sum_{a\neq 1} \mathcal{C}_a A_a B^a + [A]_1 B - [AB]_1.$$

$$\Downarrow$$

$$\boxed{\sum_{a\neq 1} \mathcal{C}_a A_a B^a = [AB]_1 - [A]_1 B \qquad \text{is connected!}}$$

- Spectral decomposition of H

$$H = \sum_{\alpha \in \mathcal{A}} |\psi_\alpha^{(-)}\rangle\langle\psi_\alpha^{(-)}|H = \sum_{\alpha \in \mathcal{A}} \sum_{a \in \mathcal{P}} [P_\alpha^{(-)} H]_a.$$

$$P_\alpha^{(-)} = \Omega^{(-)}(a)\Phi_\alpha \Phi_\alpha^\dagger (\Omega^{(-)}(a))^\dagger$$

$$(\Phi_\alpha \Phi_\alpha^\dagger)_b = ([\Phi_\alpha \Phi_\alpha^\dagger]_a)_b = 0 \qquad a \not\subseteq b$$

- Chain rule for wave operators (Kato)

$$\Omega^{(-)}(a)\Phi_\alpha = \lim_{t \to -\infty} e^{iHt} e^{-iH_a t} \Phi_\alpha =$$

$$\lim_{t \to -\infty} e^{iHt} e^{-iH_b t} e^{iH_b t} e^{-iH_a t} \Phi_\alpha = \lim_{t \to -\infty} e^{iHt} e^{-iH_b t} (\Omega^{(-)}(a))_b \Phi_\alpha =$$

$$\Omega^{(-)}(b)(\Omega^{(-)}(a))_b \Phi_\alpha.$$

for $a \subseteq b$

$$\Omega^{(-)}(a)\Phi_\alpha = \Omega^{(-)}(b)(\Omega^{(-)}(a))_b\Phi_\alpha = \Omega_b^-(a)\Phi_\alpha + (\Omega^-(a)\Phi_\alpha)^b.$$

This means

$$P_\alpha^{(-)}H = \sum_{b\neq 1}\mathcal{C}_b(P_\alpha^{(-)})_b H_b + [P_\alpha^{(-)}H]_1$$

Using this in the spectral expansion of H

$$H = \sum_{a\neq 1}\mathcal{C}_a H_a + [H]_1 = (\sum_{\alpha\in\mathcal{A}}(\sum_{b\neq 1}\mathcal{C}_b(P_\alpha^{(-)})_b H_b + [P_\alpha^{(-)}H]_1)$$

$$[H]_1 = \sum_{\alpha\in\mathcal{A}}[P_\alpha^{(-)}H]_1$$

Note $(P_\alpha^-)_b = (\Omega(a)_b^{(-)}\Phi_\alpha\Phi_\alpha^\dagger(\Omega(a)_b^{(-)})^\dagger$ uses only proper subsystem spectral projections

Channel decomposition $\mathcal{A} = \mathcal{A}_I \cup \mathcal{A}'$

$$H = \sum_{\alpha \in \mathcal{A}_I} P_\alpha^{(-)} H + \sum_{\alpha \in \mathcal{A}'} P_\alpha^{(-)} H$$

$$\sum_{\alpha \in \mathcal{A}_I} (\sum_{b \neq 1} \mathcal{C}_b(P_\alpha^-)_b H_b + [P_\alpha^- H]_1) + \sum_{\alpha \in \mathcal{A}'} (\sum_{b \neq 1} \mathcal{C}_b(P_\alpha^-)_b H_b + [P_\alpha^- H]_1)$$

$$H = \sum_{\alpha \in \mathcal{A}_I} \sum_{b \neq 1} \mathcal{C}_b(P_\alpha^-)_b H_b + \sum_{\alpha \in \mathcal{A}'} (\sum_{b \neq 1} \mathcal{C}_b(P_\alpha^{(-)})_b H_b + [H]_1.$$

$$\boxed{H_{\mathcal{A}_I} := \sum_{\alpha \in \mathcal{A}_I} \sum_{b \neq 1} \mathcal{C}_b(P_\alpha^{(-)})_b H_b}$$

$$H_{\mathcal{A}'} := \sum_{\alpha \in \mathcal{A}'} \sum_{b \neq 1} \mathcal{C}_b(P_\alpha^{(-)})_b H_b + [H]_1$$

$$H_{\mathcal{A}_I} = \sum_{\alpha \in \mathcal{A}_I} P_\alpha^{(-)} H - \sum_{\alpha \in \mathcal{A}_I} [P_\alpha^{(-)} H]_1$$

$$[P_{\mathcal{A}_I}^{(-)} H]_1 := \sum_{\alpha \in \mathcal{A}_I} [P_\alpha^{(-)} H]_1 = W_I. \qquad \text{connected}$$

$$P_\alpha^{(-)} H = H_{\mathcal{A}_I} + W$$

$$(z - H_{\mathcal{A}_I})^{-1} = (z - P_{\mathcal{A}_I}^{(-)} H)^{-1} - (z - P_{\mathcal{A}_I}^{(-)} H)^{-1} W_I (z - H_{\mathcal{A}_I})^{-1}.$$

- Differs from the resolvent of the exact projected Hamiltonian by a connected operator.

$$|\Psi_\alpha^{(-)}\rangle = \lim_{t\to-\infty} e^{iP^{(-)}_{\mathcal{A}_I}Ht} e^{-i(E_a+i0^+)t}\Phi_\alpha|\phi\rangle =$$

$$\lim_{t\to-\infty} e^{i(H_{\mathcal{A}_I}+W_I)t} e^{-i(E_a+i0^+)t}\Phi_\alpha|\phi\rangle =$$

$$\lim_{t\to-\infty} e^{i(H_{\mathcal{A}_I}+W_I)t} e^{-i(H_{\mathcal{A}_I}t} e^{iH_{\mathcal{A}_I}t} e^{-i(E_a+i0^+)t}\Phi_\alpha|\phi\rangle =$$

$$\Omega^{(-)}_{\mathcal{A}_I}(a)\Phi_\alpha|\phi\rangle + \frac{1}{E_a - P^{(-)}_{\mathcal{A}_I}H + i\epsilon} W_I \Omega^{(-)}_{\mathcal{A}_I}(a)\Phi_\alpha|\phi\rangle$$

$$|\Psi_\alpha^{(-)}\rangle_{\mathcal{A}_I} + \underbrace{\frac{1}{E_a - H_{\mathcal{A}_I} - W_I + i\epsilon} W_I \Omega^{(-)}_{\mathcal{A}_I}(a)\Phi_\alpha|\phi\rangle}_{\text{connected}}$$

- Optical theorem

$$\Phi_\beta^\dagger T_{\mathcal{A}}^{bb}\Phi_\beta = \Phi_\beta^\dagger(H_{\mathcal{A}_I}^b + H_{\mathcal{A}_I}^b(E_\beta - H_{\mathcal{A}_I} + i\epsilon)^{-1}H_{\mathcal{A}_I}^b)\Phi_\beta.$$

- Discontinuity across cut

$$\Phi_\beta^\dagger(T_{\mathcal{A}_I}^{bb}(E+i\epsilon) - T_{\mathcal{A}_I}^{bb}(E_\beta - i\epsilon))\Phi_\beta = \Phi_\beta^\dagger H_{\mathcal{A}_I}^b \frac{-2i\epsilon}{(H_{\mathcal{A}_I} - E_\beta)^2 + \epsilon^2} H_{\mathcal{A}_I}^b \Phi_\beta$$

$$= -2\pi i \Phi_\beta^\dagger H_{\mathcal{A}_I}^b \delta(H_{\mathcal{A}_I} - E_\beta) H_{\mathcal{A}_I}^b \Phi_\beta.$$

$$-2\pi i \sum_{\alpha\in\mathcal{A}_I} \Phi_\beta^\dagger H_{\mathcal{A}_I}^b \Omega^-(a)\Phi_\alpha\Phi_\alpha^\dagger \delta(E_\beta - E_\alpha)\Omega^-(a)^\dagger H_{\mathcal{A}_I}^b \Phi_\beta^\dagger.$$

$$2Im(\Phi_\beta^\dagger T^{bb}(E+i\epsilon)\Phi_\beta^\dagger) =$$

$$-2\pi \sum_{\alpha\in\mathcal{A}_I} \Phi_\beta^\dagger H^b_{\mathcal{A}_I}\Omega^{(-)}(a)\Phi_\alpha \delta(E_\beta - E_\alpha)\Phi_\alpha^\dagger \Omega^{(-)}(a)^\dagger H^b_{\mathcal{A}_I}\Phi_\beta =$$

$$-2\pi \sum_{\alpha\in\mathcal{A}} \int |\langle T^{\beta\alpha}_{\mathcal{A}_I}\rangle|^2 \delta(E_\beta - \sum_i E_{\alpha_i}) d\mathbf{p}_1 \cdots d\mathbf{p}_N$$

$$\text{RHS} = -(2\pi)\frac{v}{(2\pi)^4}\sigma_T$$

$$\sigma_T = -\frac{(2\pi)^3}{v} 2Im\Phi_\beta^\dagger T^{bb}(E+i\epsilon)\Phi_\beta^\dagger = -\frac{2(2\pi)^3\mu}{k}\frac{-1}{(2\pi)^2\mu}F_{\beta\beta} = \frac{4\pi}{k} ImF_{\beta\beta}$$

$$F_{\beta\beta} := -(2\pi)^2\mu T_{\beta\beta} \qquad \sigma_T = \frac{4\pi\mu}{k} ImF_{\beta\beta}$$

- $H_{\mathcal{A}_I}$ only supports states that asymptotically look like bound clusters in the chanels $\mathcal{A}_I$

- The scattering states of $H_{\mathcal{A}_I}$ differ from the exact scattering states when all particle as close together.

- The interactions in $H_{\mathcal{A}_I}$ can in principle be constructed from proper subsystem solutions.

- The interactions in $H_{\mathcal{A}_I}$ can in principle be constructed from proper subsystem solutions.

- Effective interactions appear with all kinds of connectivities.

- The truncated theory satisfies unitarity on the chosen important channels, $\mathcal{A}_I$. Corrections can be included systematically