Relativistic current operators

W. N. Polyzou

Jacek Gloak

The University of Iowa

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Johannes Gutenberg University

Probes of Hadronic structure

- **Short-distance electromagnetic probes of strongly interacting systems require relativistic momentum transfers, a relativistic model of the strong interaction dynamics and a consistent strong electromagnetic current. ($\Delta p \sim \hbar/\Delta x$, $\Delta x = .1$ fm $\rightarrow$ $\Delta p \sim 2$ GeV)**

- **Perturbative methods are not applicable to the strong interaction dynamics**

$$H = H_s + H_{\gamma e} + \int e \left(I_s^{\mu}(0, \mathbf{x}) + I_{e\gamma}^{\mu}(0, \mathbf{x}) \right) A_{\mu}((0, \mathbf{x}) d\mathbf{x}$$

Relativistic dynamics - one-photon-exchange approximation

$$U(\Lambda, a) \to U_s(\Lambda, a) \otimes U_{QED}(\Lambda, a)$$

$$[P_s^\mu, P_s^\nu] = 0, \qquad [J_s^i, P_s^j] = i\epsilon^{ijk} P_s^k, \qquad [J_s^i, J_s^j] = i\epsilon^{ijk} J_s^k,$$

$$[J_s^i, K_s^j] = i\epsilon^{ijk} K_s^k, \qquad [K_s^i, K_s^j] = -i\epsilon^{ijk} J_s^k$$

$$[K_s^i, P_s^j] = i\delta^{ij} H_s \qquad [K_s^i, H_s] = iP_s^i.$$

Consistent current covariance and conservation

$$[K_s^i, I_s^j(0)] = i\delta^{ij} I_s^0(0), \qquad [K_s^i, I_s^0(0)] = iI_s^i(0)$$

$$[H_s, I_s^0(0)] = \sum_i [P_s^i, I_s^i(0)]$$

Cluster properties

$$H_s = \sum_i H_i + \sum H_{sij} + \sum H_{sijk} + \cdots$$

$$\mathbf{K}_s = \sum_i \mathbf{K}_{si} + \sum \mathbf{K}_{sij} + \sum \mathbf{K}_{sijk} + \cdots$$

$$I_s^\mu(0) = \sum_i I^\mu(0)_{si} + \sum I^\mu(0)_{sij} + \sum I^\mu(0)_{sijk} + \cdots$$

One-photon exchange approximation
Gell Mann - Goldberger two-potential formula

Approximate transition matrix elements

$$\langle F|T|I\rangle = \int d\mathbf{x}d\mathbf{y}\langle F_s|\Omega^\dagger_{s+}I^\mu_s(0,\mathbf{x})\Omega_{s-}|I_s\rangle \times$$

$$\langle 0|T(A_\mu(0,\mathbf{x})A_\mu(0,\mathbf{y}))|0\rangle\langle e'|I^\nu_e(0,\mathbf{y})|e\rangle$$

$$\Omega_{s\pm} = \lim_{t\to\pm\infty} e^{iH_s t}\Pi e^{-iH_0 t} \qquad \text{or bound state}$$

Scattering Equivalences - change of representation

$$H'_s = W H_s W^\dagger$$

$$S_s = \Omega^\dagger_{s+} \Omega_{s-}$$

$$\Omega_{s\pm} = s - \lim_{t \to \pm\infty} e^{iH_s t} \Pi e^{-iH_0 t} \qquad \Omega'_{s\pm} = s - \lim_{t \to \pm\infty} e^{iH'_s t} \Pi' e^{-iH_0 t}$$

$$S_s = S'_s \iff W^\dagger W = I$$

and

$$s - \lim_{t \to \pm\infty} (W\Pi - \Pi') e^{-iH_0 t} = 0 \qquad \textbf{both time limits!}$$

Models and many-body strong currents are representation dependent.

For translationally and rotationally invariant W

$$[W, \mathbf{P}_s] = [W, \mathbf{J}_s] = 0$$

a consistent calculation requires

$$H_s' = W H_s W^\dagger$$

$$\mathbf{K}_s' = W \mathbf{K}_s W^\dagger$$

$$I_s^{\mu\prime}(0) = W I_s^{\mu}(0) W^\dagger$$

Given a relativistic model of strongly interacting particles

- **The impulse current is not consistent with the dynamics**

- **It is possible to compute independent current matrix elements and use current covariance and current conservation to calculate the the remaining current matrix elements. The results will depend on the choice of independent matrix elements. In addition this method cannot be consistently applied to different reactions.**

- **Is it possible to construct a strong current operator that is consistent with the relativistic dynamics?**

Weyl representation

Irreducible set of operators $\{\hat{\mathbf{q}}_i, \hat{\mathbf{p}}_i\}$

Any operator, $\hat{H}$, (local or non-local) can be expressed in the form:

$$\boxed{\hat{H} = \int d^{3N}\mathbf{a}\, d^{3N}\mathbf{b}\, h(\mathbf{a}, \mathbf{b}) e^{i\mathbf{a}\cdot\hat{\mathbf{q}}} e^{i\mathbf{b}\cdot\hat{\mathbf{p}}}}$$

where

$$\hat{\mathbf{p}} = (\hat{p}_1, \cdots, \hat{p}_N)$$

$$\hat{\mathbf{q}} = (\hat{q}_1, \cdots, \hat{q}_N)$$

$$[\hat{q}_i, \hat{p}_j] = i\delta_{ij}.$$

Relativistic case - irreducible operators $\{\mathbf{q}_i, \mathbf{p}_i\}$

$\mathbf{q}_i =$ Newton-Wigner position operator

function of single-particle Poincaré generators

$$\mathbf{q}_i := -\frac{1}{2}\{\frac{1}{H_i}, \mathbf{K}_i\} - \frac{\mathbf{P}_i \times (H_i \mathbf{J}_i - \mathbf{K}_i)}{M_i H_i (M_i + H_i)} = i\boldsymbol{\nabla}_p$$

where the partial derivative is computed holding the canonical spin constant (recall Wigner rotations are momentum dependent).

$$[q_i, p_j] = -i\delta_{ij}$$

Define the strong current using local gauge invariance

Steps

- Represent the relativistic Hamiltonian in the Weyl representation.
- Replace the operators $\mathbf{p}_i$ in the Weyl representation of the Hamiltonian by gauge covariant derivatives.
- Extract the term linear in the vector potential
- Identify the current with the coefficient of the vector potential
- Factorization requires dealing with non-commuting operators.

$$\hat{H}_S \to \hat{H}_s = \int d^{3N}\mathbf{a} d^{3N}\mathbf{b} h(\mathbf{a},\mathbf{b}) e^{i\mathbf{a}\cdot\hat{\mathbf{q}}} e^{i(\hat{\mathbf{p}}-e\hat{A}(\hat{\mathbf{q}}))\cdot\mathbf{b}}$$

The term linear in $\hat{\mathbf{A}}(\mathbf{q})$ (use Trotter product formula to get)

$$e\frac{d\hat{H}}{de}_{|_{e=0}} =$$

$$-\int_0^1 d\lambda \int d^{3n}\mathbf{a} d^{3n}\mathbf{b} h(\mathbf{a},\mathbf{b}) e^{i\mathbf{a}\cdot\hat{\mathbf{q}}} e^{i\lambda\hat{\mathbf{p}}\cdot\mathbf{b}} \sum_j \hat{\mathbf{A}}(\hat{\mathbf{q}}_j)\cdot\mathbf{b}_j e^{i(1-\lambda)\hat{\mathbf{p}}\cdot\mathbf{b}}$$

Problem of non-commuting operators

Since $\mathbf{p}_i$ generates translations use

$$e^{i\lambda\hat{\mathbf{p}}\cdot\mathbf{b}}\mathbf{q}_i e^{-i\lambda\hat{\mathbf{p}}\cdot\mathbf{b}} = (\mathbf{q}_i + \lambda\mathbf{b}_i)$$

to get:

$$e^{i\lambda\hat{\mathbf{p}}\cdot\mathbf{b}} \sum_j \hat{\mathbf{A}}(\hat{\mathbf{q}}_j) \cdot \mathbf{b}_j e^{i(1-\lambda)\hat{\mathbf{p}}\cdot\mathbf{b}} =$$

$$\sum_j \hat{\mathbf{A}}(\hat{\mathbf{q}}_j + \lambda\mathbf{b}_j) \cdot \mathbf{b}_j e^{i\hat{\mathbf{p}}\cdot\mathbf{b}}$$

This puts the $\mathbf{q}_i$ dependence to the left of the $\mathbf{p}_i$ dependence:

$$e\frac{d\hat{H}}{de}_{|e=0} =$$

$$-\int_0^1 d\lambda \int d^{3n}\mathbf{a}d^{3n}\mathbf{b}h(\mathbf{a},\mathbf{b})e^{i\mathbf{a}\cdot\hat{\mathbf{q}}}\sum_j \hat{\mathbf{A}}(\hat{\mathbf{q}}_j+\mathbf{b}_j)\cdot\mathbf{b}_j e^{i\hat{\mathbf{p}}\cdot\mathbf{b}}$$

Evaluate in the mixed representation to replace operators by a complex kernel

$$e\langle \mathbf{q}_1 \cdots \mathbf{q}_N | \frac{d\hat{H}}{de}_{|_{e=0}} | \mathbf{p}_1 \cdots \mathbf{p}_N \rangle =$$

$$-e \int_0^1 d\lambda \int \frac{d^{3N}\mathbf{a} d^{3N}\mathbf{b}}{(2\pi)^{3N/2}} h(\mathbf{a}, \mathbf{b}) e^{i\mathbf{a}\cdot\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{p}} \sum_j \hat{\mathbf{A}}(\mathbf{q}_j + \lambda \mathbf{b}_j) \cdot \mathbf{b}_j e^{i\mathbf{p}\cdot\mathbf{b}}$$

Fourier transform to get momentum space matrix elements

$$e\langle \mathbf{p}'_1 \cdots \mathbf{p}'_n | \frac{d\hat{H}}{de}_{|_{e=0}} | \mathbf{p}_1 \cdots \mathbf{p}_n \rangle =$$

$$-e \int_0^1 d\lambda \int \frac{d^{3N}\mathbf{a} d^{3N}\mathbf{b} d^{3N}\mathbf{q}}{(2\pi)^{3N}} h(\mathbf{a}, \mathbf{b}) e^{-i\mathbf{q}\cdot(\mathbf{p}'-\mathbf{p}-\mathbf{a})} \sum_j \hat{\mathbf{A}}(\mathbf{q}_j + \lambda \mathbf{b}_j) \cdot \mathbf{b}_j e^{i\mathbf{p}\cdot\mathbf{b}}$$

Factorization

Use

$$\hat{\mathbf{A}}(\mathbf{q}_j + \lambda \mathbf{b}_j) = \int d\mathbf{q}' \hat{\mathbf{A}} \delta(\mathbf{q}_j + \lambda \mathbf{b}_j - \mathbf{q}')) \hat{\mathbf{A}}(\mathbf{q})$$

to factor the vector potential

$$e\langle \mathbf{p}'_1 \cdots \mathbf{p}'_n | \frac{d\hat{H}}{de}_{|_{e=0}} | \mathbf{p}_1 \cdots \mathbf{p}_n \rangle =$$

$$-e \int_0^1 d\lambda \int \frac{d^{3N}\mathbf{a} d^{3N}\mathbf{b} d^{3N}\mathbf{q} d\mathbf{q}'}{(2\pi)^{3N}} h(\mathbf{a},\mathbf{b}) e^{-i\mathbf{q}\cdot(\mathbf{p}'-\mathbf{p}-\mathbf{a})} e^{i\mathbf{p}\cdot\mathbf{b}} \times$$

$$\sum_j \delta(\mathbf{q}_j + \lambda \mathbf{b}_j - \mathbf{q}')) \mathbf{b}_j \cdot \hat{\mathbf{A}}(\mathbf{q}')$$

Expression for vector part of the current

$$\langle \mathbf{p}_1' \cdots \mathbf{p}_n' | \mathbf{J}(\mathbf{q},0) | \mathbf{p}_1 \cdots \mathbf{p}_n \rangle =$$

$$-e \int_0^1 d\lambda \int \frac{d^{3N}\mathbf{a} d^{3N}\mathbf{b} d^{3N}\mathbf{q}'}{(2\pi)^{3N}} h(\mathbf{a},\mathbf{b}) e^{-i\mathbf{q}'\cdot(\mathbf{p}'-\mathbf{p}-\mathbf{a})} e^{i\mathbf{p}\cdot\mathbf{b}} \times$$

$$\sum_j \delta(\mathbf{q}_j' + \lambda \mathbf{b}_j - \mathbf{q}) \mathbf{b}_j .$$

Covariance gives the charge density

$$J^0(\mathbf{q},0) = i[K^i, J^i(\mathbf{q},0)] \qquad \textbf{no sum, any } i$$

The relativistic kinetic energy has the form

$$H = \sqrt{\mathbf{p}^2 + m^2}$$

The above method gives

$$\langle \mathbf{p}' | \mathbf{J}(\mathbf{x}, 0) | \mathbf{p} \rangle =$$

$$-e \frac{1}{(2\pi)^3} \int_0^1 d\lambda \frac{(1-\lambda)\mathbf{p} + \lambda \mathbf{p}'}{\sqrt{((1-\lambda)\mathbf{p} + \lambda \mathbf{p}')^2 + \mathbf{m}^2}} e^{i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')}$$

Compared to non-relativistic result $(H = \frac{\mathbf{p}^2}{2m})$**:**

$$\langle \mathbf{p}' | \mathbf{J}(\mathbf{x}, 0) | \mathbf{p} \rangle = -e \frac{1}{(2\pi)^3} \frac{\mathbf{p}' + \mathbf{p}}{2m} e^{i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')}$$

Remarks - conclusions

- Assumes particles are point charges. Modifications needed for charge distributions.
- Result is operators rather than matrix elements - in principle gives applicable to different reactions.
- Consistent with the dynamics - base on a gauge invaraint Hamiltoninan.
- Method applicable to non-local interactions
- Light-front version - uses a different representation of the Weyl algebra.
- Charge density operator requires a representation of the dynamical boost generator (otherwise can use current conservation in matrix elements)
- Test application - applied to NR spin orbit and L^2 parts of V18, assuming point charges. results too small to impact calculation of A, B and tensor polarzation in elastic electron deuteron sacttering?

Thanks - organizers!