Simple relativistic models of hadrons

W. N. Polyzou

Siraj Kuthini Kunhammed, Jonatan Posigula, Mark Tucker

The University of Iowa

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Goal:

Construct (front-form) relativistic models of hadrons motivated by QCD.

Elements:

QCD degrees of freedom (locally and globally SU(3) invariant).

Scales set by quark masses, 1 coupling constant, CSB scale.

Simple enough to treat sea quark degrees of freedom.

Dual QCD - hadronic representations.

Consistent treatment of scattering, decays, spectra and em properties.

Justification:

Direct QCD treatment of hadronic reactions is difficult; however QCD is a powerful tool for deriving and constraining the structure of degrees of freedom and interactions in models of QCD.

Inspiration:

- Structure of the model (degrees of freedom/interactions):

    K. G. Wilson, Phys. Rev. D10, 2445 (1974).

    J. B. Kogut and L. Susskind, Phys. Rev. D11, 395 (1975).

    E. Seiler, Lecture Notes in Physics, 159, 1 (1982).

- Treatment of glue DOF

    O. W. Greenberg and J. Hietarinta, Physics Letters B 86, 309 (1979).

- Scattering in confined systems

    R. F. Dashen, J. B. Healy, and I. J. Muzinich, Ann. of Phys. 102, 1 (1976).

Model Hilbert space - motivation for structure:

Kogut and Susskind: (Hamiltonian lattice) degrees of freedom are mutually non-interacting global and local SU(3) color invariant connected networks of quarks, anti-quarks and links.

The static energy of a connected network is equal to the sum of the quark masses and the number of links times the energy per link.

K & S Hilbert space: locally and globally gauge invariant eigenstates of quark and anti-quark masses and link energy.

In the absence of the remaining interactions the static degrees of freedom are confined. Local gauge invariance means separating quarks requires more links.

Model Hilbert space

Model connected local and global color singlets by confined systems of quarks and anti-quarks. In general there will be towers of excited interactions.

Greenberg and Hietarinta: Identical quarks in different connected networks behave like distinguishable particles due to the link degree of freedom.

$$\langle \downarrow\uparrow \mid \rightleftarrows \rangle = \langle 0 | \Box \rangle = 0$$

$\rightarrow$ Quarks and anti-quarks confined in different connected singlets are treated as distinguishable. This eliminates Van der Waals forces.

The Hilbert space has a dual representation as space of bare confined singlets with hadronic quantum numbers.

Dynamics

Covariant derivative and color magnetic interactions allow different connected singlets to move and interact.

Too many gauge invariant degrees of freedom and too many interactions between them.

Dynamical Assumption: The physics involving the lowest energy degrees of freedom is dominated by string breaking and the “ground” confining interaction.

No fundamental QCD justification, except that meson exchange seems to be important in hadronic reactions and string breaking is used successfully to model hadronic reactions in PYTHIA.

Question: Given this limitation on the degrees of freedom and interactions do we get a consistent picture of spectral properties, lifetimes, cross sections and electromagnetic observables using a limited set of parameters?

Model - meson valence sector:

Mass operator for a quark-anti-quark pair - scales set by model parameters:

$$M_c = \sqrt{k^2 + V_c + m_q^2} + \sqrt{k^2 + V_c + m_{\bar{q}}^2}$$

$$\boxed{V_c = -\frac{\lambda^2}{4}\nabla_k^2 + V_0}$$

$$M_{nl} \to \sqrt{m_q^2 + \lambda(2n + l + \frac{3}{2}) + V_0} + \sqrt{m_{\bar{q}}^2 + \lambda(2n + l + \frac{3}{2}) + V_0}.$$

π and $\pi - \rho$ splitting

$$V_{csb} := (a + b\mathbf{s}_q \cdot \mathbf{s}_{\bar{q}})\delta_{l0}.$$

V_0 and the quark masses are essentially the same parameter. This is an arbitrary splitting of a single constant. There are no quark mass eigenstates - there is no way to separate what we call a quark mass from what we call a confining interaction.

We simply assume that quarks and anti-quarks transform like mass m_q spin $\frac{1}{2}$ irreducible representations of the Poincaré group (no fundamental justification).

Bare mesons

Approximate linear confinement

$$\langle r_{nls}^2 \rangle^{1/2} = \sqrt{\frac{2}{\lambda}(2n + l + \frac{3}{2})} \qquad M_{nls} \approx \sqrt{2}\lambda \langle r_{RMS}^2 \rangle^{\frac{1}{2}}$$

Approximate Regge behavior

$$l \approx \frac{1}{4\lambda} M_{nls}^2$$

The oscillator parameter is chosen to fit the Regge slope of the $\rho - a$ mesons.

Table: Regge trajectories, $J = L + 1, S = 1$ $m_q = \frac{m_\rho}{2} = .385$, $\lambda = .282$

meson	L	exp mass	exp $(\text{mass})^2$	J	calc mass	calc $(\text{mass})^2$
ρ	0	.770	.593	1	.770	.593
a_2	1	1.320	1.742	2	1.311	1.719
ρ_3	2	1.690	2.856	3	1.687	2.846
a_4	3	2.040	4.162	4	1.994	3.976
ρ_5	4	2.350	5.522	5	2.259	5.103
a_6	5	2.450	6.000	6	2.497	6.335

$\langle r_\pi^2 \rangle^{1/2} = .64$ fm

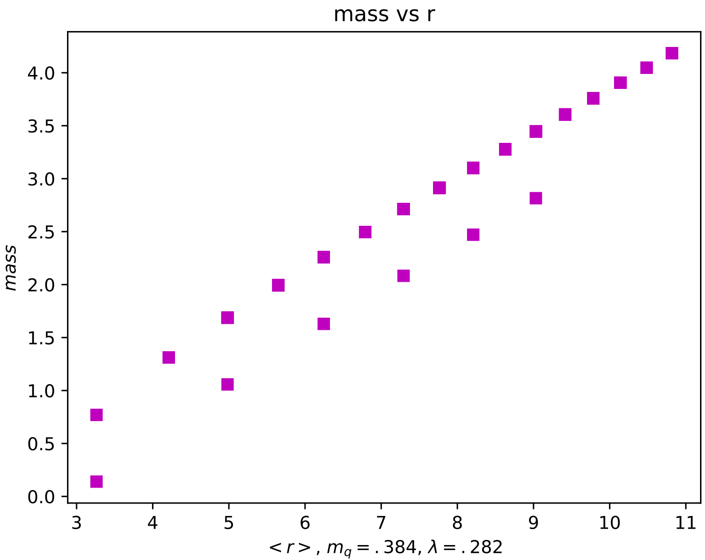


Figure: mass vs $\langle r^2 \rangle^{1/2}$

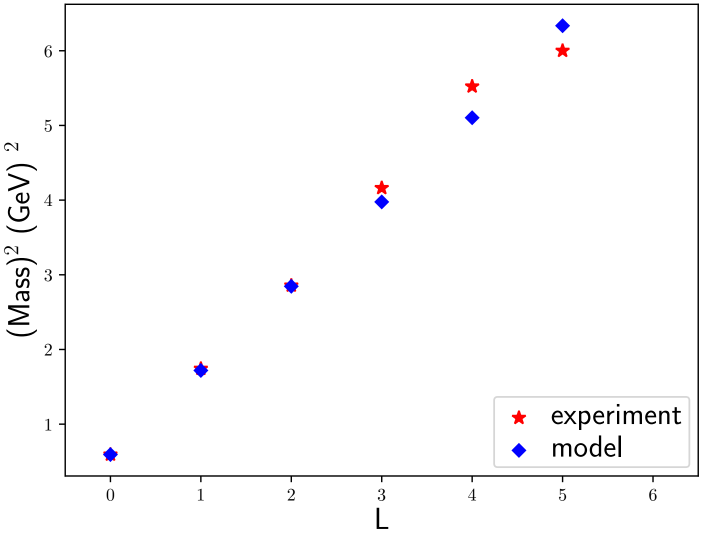


Figure: Regge trajectory for ρ and a mesons

Relativity (unitary representation of the Poincaré group

Relative momentum relativistic:

$$\langle k_{nls}^2 \rangle^{1/2} = \sqrt{\frac{\lambda}{2}(2n + l + \frac{3}{2})}. \qquad \sqrt{\frac{3\lambda}{4}} \approx .46(GeV)$$

Bare hadron wave functions:

$$\langle \tilde{\mathbf{P}}, j, \tilde{\mu}, k, l, s | \tilde{\mathbf{P}}', j', \tilde{\mu}', n', l', s' \rangle = \delta(\tilde{\mathbf{P}} - \tilde{\mathbf{P}}') \delta_{\tilde{\mu}\tilde{\mu}'} \delta_{j'j} \delta_{s's} \delta_{l'l} \tilde{R}_{n'l'}(k).$$

Hadronic (dual) representation of Hilbert space:$(k \leftrightarrow n)$

$$\mathcal{H}_H := \oplus \mathcal{H}_{njls}.$$

Unitary representation of the Poincaré group on $\mathcal{H}_H$

$$U_H(\Lambda, a) = \sum_{njls} U_{njls}(\Lambda, a)$$

$$U_{njls}(\Lambda, a) | \tilde{\mathbf{P}}, j, \tilde{\mu}, n, l, s \rangle =$$

$$e^{-ia \cdot \Lambda P_{nls}} \sum_{\tilde{\nu}} | \tilde{\mathbf{\Lambda}} P_{nls}, j, \tilde{\nu}, n, l, s \rangle \sqrt{\frac{(\Lambda P_{nls})^+}{P^+}} D^j_{\tilde{\nu}\tilde{\mu}} [B_f^{-1}(\Lambda P_{nls}) \Lambda B_f(P_{nls})]$$

Summary - bare hadrons

Wave functions are known analytically (harmonic oscillator).

Unitary representation of the Poincaré group - including transverse rotations.

Approximate linear confinement.

Approximate linear Regge trajectory - slope fixes λ.

Only flavor dependence is quark masses at this point.

Gauge invariant basis.

String breaking

A quark-anti-quark pair is produced with equal probability at any point on the line between the original quark-anti-quark pair.

Delta functions are replaced by delta-function normalized Gaussians with the width of oscillator ground state (replaces line by a "flux tube" with width determined by oscillator parameter λ).

Spin independent vertex:

$$\langle \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_{12} | v_{2:1} | \mathbf{r} \rangle := g\sqrt{\lambda}\delta(\mathbf{r} - 2\mathbf{r}_{12}) \int_0^1 d\eta \delta_{\sqrt{\frac{\lambda}{2}}}(\mathbf{r}_1 - \eta\mathbf{r})\delta_{\sqrt{\frac{\lambda}{2}}}(\mathbf{r}_2 - (1-\eta)\mathbf{r})$$

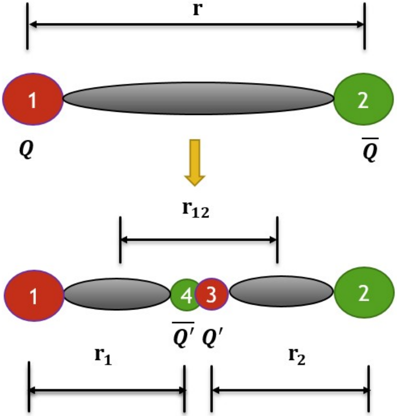
where the Gaussian approximate delta function is

$$\delta_{\sqrt{\frac{\lambda}{2}}}(\mathbf{r}) := (\frac{\lambda}{4\pi})^{3/2} e^{\frac{-\lambda r^2}{4}} \qquad \int \delta_{\sqrt{\frac{\lambda}{2}}}(\mathbf{r}) d\mathbf{r} = 1.$$

The dimensionless coupling constant g must be a constant of order unity.

Spin dependent part (q,$\bar{q}$ have opposite parity):

$$Y_{1m}(\hat{\mathbf{r}}_{12})\langle s_3, \mu_3, s_4, \mu_4 | 1, \mu_s \rangle \langle 1, m_l, 1, \mu_s | 0, 0 \rangle.$$



r
1
2
Q
$\overline{Q}$
r_{12}
1
4
3
2
$\overline{Q'}$ Q'
r_1
r_2

Hadronic representation of vertex (spin independent part)

The 9 dimensional integral over the initial and final bare meson states can be computed analytically for any three bare meson states

The string breaking vertex fixes all hadronic production vertices:

$$\langle n_1, l_1, m_1, n_2, l_2, m_2, \mathbf{r}_{12} | v_{2:1} | n, l, m \rangle =$$

$$\frac{g}{\sqrt{\lambda}} R_{nl}(2r_{12})(2\lambda)^{3/2} \frac{(\sqrt{\frac{\lambda}{2}} r_{12})^{2n_1+l_1+2n_2+l_2}}{\sqrt{2n_1!\Gamma(n_1+l_1+\frac{3}{2})}\sqrt{2n_2!\Gamma(n_2+l_2+\frac{3}{2})}} \times$$

$$e^{-\frac{\lambda}{4}r_{12}^2} \sum_{k_1+k_2=2r} \frac{(l_1+2n_1)!(l_2+2n_2)!}{k_1!k_2!(l_1+2n_1-k_1)!(l_2+2n_2-k_2)!} (-)^{k_2} (\frac{1}{2})^{l_1+2n_1+l_2+2n_2} \times$$

$$\frac{1}{2r+1} M(\frac{1}{2}+r, \frac{3}{2}+r, -\frac{\lambda r_{12}^2}{4}) Y_{lm}(\hat{\mathbf{r}}_{12}) Y^*_{l_1 m_1}(\hat{\mathbf{r}}_{12}) Y^*_{l_2 m_2}(\hat{\mathbf{r}}_{12}).$$

Momentum space requires a one-dimensional Fourier Bessel transform of r_{12}.

The full vertex is defined by embedding it in the full Hilbert space so it commutes with and is independent of P^+, $\mathbf{P}_\perp$ and $\mathbf{S}_f$.

Tweaks:

The structure of the model is constrained because the scales are essentially fixed by one parameter.

We had trouble getting a consistent picture of scattering, lifetimes, bare meson spectra due to these constraints.

This was fixed by applying a unitary scale transformation to the vertex that reduced the width of the flux tube by a factor of 2.

$$\langle n_1, l_1, m_1, n_2, l_2, m_2, \mathbf{r}_{12}|v_{2:1}|n, l, m\rangle \to (2)^{3/2}\langle n_1, l_1, m_1 n_2, l_2, m_2, 2r_{12}|v_{2:1}|n, l, m\rangle$$

This is still consistent with the scale set by the confining interaction.

The up and down quark masses were taken to be half of the ρ mass. The only calculations sensitive to the quark masses were the form factor calculations. Pion form factor calculations ignoring sea quarks were closer to data using m_q : .385 GeV$\to$ .2 GeV. These calculations did not include sea quark contributions.

Sea quarks - truncation to 1+2 bare meson subspace

Model Hilbert space

$$\mathcal{H} = \mathcal{H}_H \oplus (\mathcal{H}_H \otimes \mathcal{H}_H) \qquad \text{Hadronic representation.}$$

$$\mathcal{H} = \mathcal{H}_{q\bar{q}} \oplus (\mathcal{H}_{q\bar{q}} \otimes \mathcal{H}_{q\bar{q}}) \qquad \text{QCD DOF representation.}$$

Bare meson unitary representation of the Poincaré group

$$U_0(\Lambda, a) = \begin{pmatrix} U(\Lambda, a) & 0 \\ 0 & U(\Lambda, a) \otimes U(\Lambda, a) \end{pmatrix}.$$

String breaking dynamics

$$M = M_0 + V = \underbrace{\begin{pmatrix} M_c & 0 \\ 0 & \sqrt{M_{c1}^2 + \mathbf{q}^2} + \sqrt{M_{c2}^2 + \mathbf{q}^2} \end{pmatrix}}_{M_0} + \underbrace{\begin{pmatrix} 0 & v_{1:2} \\ v_{2:1} & 0 \end{pmatrix}}_{V},$$

$v_{i:j}$ is the string breaking vertex.

Dynamical Relativity

String breaking vertex constructed to commute with light-front kinematic subgroup and $\mathbf{s}_{f0}$ (not $\mathbf{J}_0$!).

Diagonalize M in the basis of simultaneous eigenstates of $M_0, P_0^+, \mathbf{P}_{0\perp}, s_0^2, s_{0fz}$ and invariant degeneracy quantum numbers, d.

$U(\Lambda, a)$ is defined so these states transform irreducibly

$$U(\Lambda, a)|(M, s, d)\tilde{\mathbf{P}}, \tilde{\mu}\rangle :=$$

$$e^{-ia\cdot\Lambda P_M} \sum_{\tilde{\nu}} |(M, s, d)\tilde{\mathbf{\Lambda}}, P_M, \tilde{\nu}\rangle \sqrt{\frac{(\Lambda P)^+}{P^+}} D^s_{\tilde{\nu}\tilde{\mu}}[B_f^{-1}(\Lambda P_M)\Lambda B_f(P_M)]$$

This is different than $U_0(\Lambda, a)$. It requires diagonalizing M. The operators $M, P_0^+, \mathbf{P}_{0\perp}, s_0^2, s_{0fz}$ are commuting self-adjoint operators. $U(\Lambda, a)$ is defined so simultaneous eigenstates of these operators transform irreducibly.

Mass eigenvalue problem:

$$|\Psi\rangle = \begin{pmatrix} |\Psi_1\rangle \\ |\Psi_2\rangle \end{pmatrix}$$

Coupled eigenvalue equations

$$(\lambda - M_c)|\Psi_1\rangle = v_{1:2}|\Psi_2\rangle$$

$$(\lambda - \sqrt{M_{c1}^2 + \mathbf{q}^2} + \sqrt{M_{c2}^2 + \mathbf{q}^2})|\Psi_2\rangle = v_{2:1}|\Psi_1\rangle$$

These decouple

$$|\Psi_1\rangle = (\lambda - M_c)^{-1} v_{12}(\lambda - \sqrt{M_{c1}^2 + \mathbf{q}^2} + \sqrt{M_{c2}^2 + \mathbf{q}^2}))^{-1} v_{2:1}|\Psi_1\rangle$$

$$|\Psi_2\rangle = (\lambda - \sqrt{M_{c1}^2 + \mathbf{q}^2} + \sqrt{M_{c2}^2 + \mathbf{q}^2}))^{-1}|\Psi_1\rangle$$

Normalization

$$1 = \langle\Psi_1|\Psi_1\rangle + \langle\Psi_2|\Psi_2\rangle$$

$\langle\Psi_2|\Psi_2\rangle$ = sea quark probability

Equation still has an infinite number of channels - it requires a truncation.

Mass eigenvalues are real zeroes of $F(\lambda)$ between 0 and the two meson threshold:

$$F(\lambda) = \det\left(I - (\lambda - M_c)^{-1} v_{1:2}(\lambda - \sqrt{M_{c1}^2 + \mathbf{q}^2} + \sqrt{M_{c2}^2 + \mathbf{q}^2}))^{-1} v_{2:1}\right).$$

Results:

Model calculation keeping 2 $q\bar{q}$ channels with n $\leq$ 4:

Table: Parameters

λ	.282 $(\mathrm{GeV})^2$
g	5.44
$m_q = m_{\bar{q}}$	.385 GeV
m_{π_0}	.160 GeV
$m_{\rho 0}$	.882 GeV

Table: Results

bare pion mass	.1600 GeV
m_π - 2^{nd} order perturbation theory ($n \leq 4$)	.1327 GeV
m_π exact ($n \leq 4$)	.1329 GeV
valence quark probability	82%
sea quark probability	16%

Scattering of bare mesons:(*s*-channel case)

Time-dependent methods result in coupled equations

$$T^{22}(e+i0^+) = 0 + v_{2:1}(e-M_1+i0^+)^{-1}T^{12}(e+i0^+)$$

$$T^{12}(e+i0^+) = v_{1:2} + v_{1:2}(e-M_2+i0^+)^{-1}T^{22}(e+i0^+).$$

These equations can expressed in terms of the solution of

$$T^{12}(e+i0^+) = v_{1:2} + v_{1:2}(e-M_2+i0^+)^{-1}v_{2:1}(e-M_1+i0^+)^{-1}T^{12}(e+i0^+).$$

This equation has an infinite number of poles in the continuum. These are spurious and can be eliminated by defining

$$\Gamma_{12}(e+i0^+) := (e-M_1+i0^+)^{-1}T^{12}(e+i0^+)$$

$$\Gamma_{12}(e+i0^+) = (e-M_1-v_{1:2}(e-M_2+i0^+)^{-1}v_{2:1})^{-1}v_{1:2}$$

$$T^{22}(e+i\epsilon^+) = v_{2:1}\frac{1}{e-M_1-v_{1:2}(e-M_2+i0^+)^{-1}v_{2:1}}v_{1:2}$$

This has no spurious singularities in the continuum.

Note that there are no long-range Van der Waals forces because the quarks in different singlets are treated as distinguishable.

Data: Phys. Rev. D7,1279(1973), Phys. Rev. D12,681(1975).

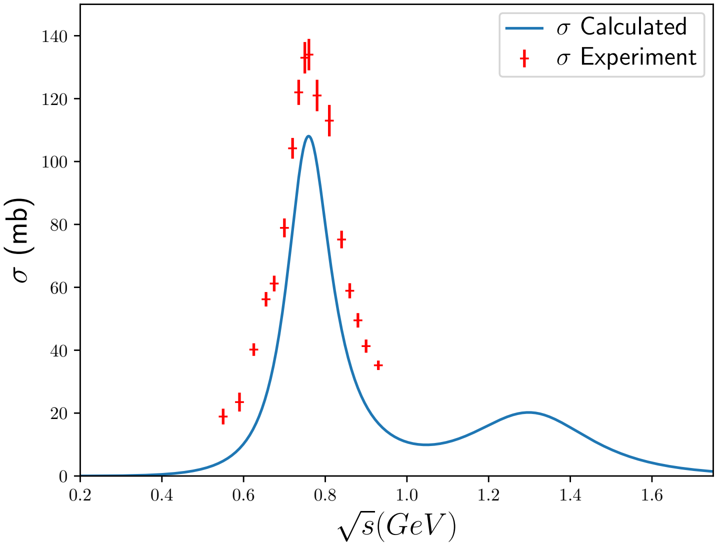


Figure: s-channel $\pi - \pi$ scattering cross section

Unstable particles

When

$$M_{n_1,n_2,0} < M_{n_0}$$

$M_{n_1,n_2,q_{120}} = M_{n_0}$ has solutions for real q_{120}^2 that depend on n_1 and n_2:

$$q_{120}^2 = \frac{M_{n_1}^4 + M_{n_2}^4 + M_{n_0}^4 - 2M_{n_1}^2 M_{n_2}^2 - 2M_{n_1}^2 M_{n_0}^2 - 2M_{n_2}^2 M_{n_0}^2}{4M_{n_0}^2}$$

The decay width is

$$\Gamma = \sum_{\mathbf{n}_1 \mathbf{n}_2} 2\pi \frac{q_{120}\omega_{n1}(q_{120})\omega_{n2}(q_{120})}{\omega_{n1}(q_{120}) + \omega_{n2}(q_{120})} |\langle n_1, n_2, q_{120}|v_{21}|n_0\rangle|^2$$

The sum is over the open decay channels.

Table: Results

bare ρ mass	.882 GeV
position ρ resonance (fixes g)	.770 GeV
shift	-.122 GeV
calculated width of ρ resonance	.134 GeV
experimental width of ρ resonance	.150 GeV

Pion Form factor - including sea quark contributions

$$F_\pi(Q^2) = \langle \pi, \tilde{\mathbf{p}}' | I^+(0) | \pi, \tilde{\mathbf{p}} \rangle$$

$$F_\pi(Q^2) =$$
$${}_1\langle \pi, \tilde{\mathbf{p}}' | I^\mu(0) | \pi, \tilde{\mathbf{p}} \rangle_1 +$$
$${}_1\langle \pi, \tilde{\mathbf{p}}' | I^\mu(0) | \frac{1}{m_\pi - M_2} v_{2:1} \frac{1}{m_\pi - M_1} | \pi, \tilde{\mathbf{p}} \rangle_1 +$$
$${}_1\langle \pi, \tilde{\mathbf{p}} | \frac{1}{m_\pi - M_1} v_{12} \frac{1}{m_\pi - M_2} | I^\mu(0) | \pi, \tilde{\mathbf{p}} \rangle_1 +$$
$${}_1\langle \pi, \tilde{\mathbf{p}} | \frac{1}{m_\pi - M_1} v_{12} \frac{1}{m_\pi^* - M_2} | I^\mu(0) | \frac{1}{m_\pi^* - M_2} v_{2:1} \frac{1}{m_\pi - M_1} | \pi, \tilde{\mathbf{p}}' \rangle_1$$

Calculations below do not include sea quark contribution

FF data from: Nuclear Physics B 277, 168 (1986), Phys. Rev. Lett. 86, 1713 (2001), Phys. Rev. D 17, 1693 (1978)

cp: Phys. Rev. C 71, 028202 (2005).

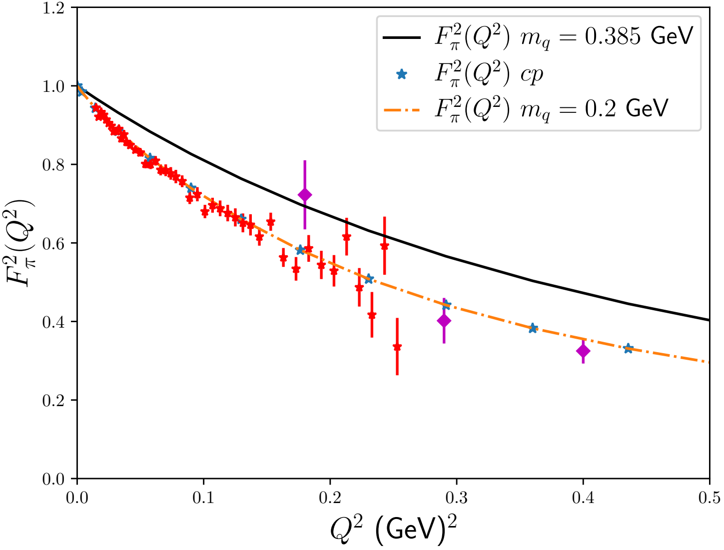


Figure: Pion Form Factor

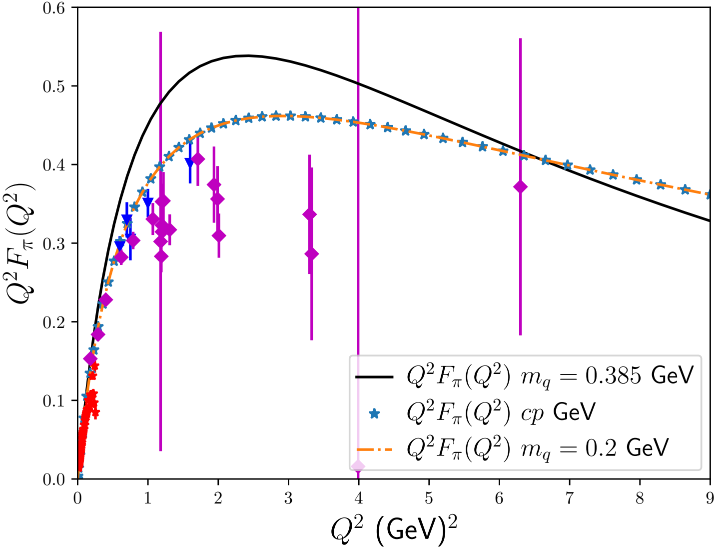


Figure: Pion Form Factor

Conclusions/Outlook:

- Simple models with the same # of parameters as QCD and dynamics given by string breaking gives a qualitatively consistent picture of spectral properties, lifetimes, cross sections and electromagnetic properties.

- Predicts sea quark properties (about 16%) for pion.

- (Posigula - UI undergraduate) Similar calculations treating baryons as a confined singlet consisting of a quark and di-quark using the same parameters gives a baryon spectrum that qualitatively agrees with experiment. Predicts percentage of sea quarks in the proton (did not include spin dependent part of vertex).

To do:

- Treatment of exotics using di-quarks.

- Include spin dependent vertex in baryon calculations.

- Nucleon and pion form factors including contribution from sea quarks.

- Include flavor dependence.

- Calculate distribution functions including sea quark DOF, etc.