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Light-front quantum field theory
Advantages:

- **Hamiltonian formulation - non-perturbative calculations are reduced to linear algebra.**
- **Light-front preserving boosts are kinematic.**
- **Light-front boosts form a closed subgroup - light-front spins do not Wigner rotate.**
- **Frame-independent impulse approximation.**
- **Vacuum is essentially trivial.**
- **Fields restricted to the light front are irreducible.**
- **7 parameter (largest) kinematic subgroup - three translations, three boosts, one rotation.**

Light-front quantum field theory compared to canonical or covariant QFT

- The problem of inequivalent representations of the canonical commutation relations.
- The problem of the trivial vacuum.
- The problem of the ill-posed initial value problem.
- The problem of rotational covariance.
- The problem of zero modes.
- The problem spontaneously broken symmetries.

Brief summary of conclusions

- Each theory has one true (dynamical) vacuum.
- The Wightman functions can be expressed as vacuum expectation values of operators restricted to a light front, using the vacuum vector of any theory.
- The dynamical content of different theories is contained in a light-front sub-algebra rather than the vacuum.
- The dynamical sub-algebra of LF operators is causal.
- Rotational invariance and space reflection symmetry are not encoded in P^- and the kinematic generators.
- SSB charges on the light-front have no dynamical content; Fixed-time conditions for Goldstone bosons can be expressed in terms of the dynamical light-front sub-algebra.
- The rest of this talk will be used to clarify these remarks.

Background - Field Algebras

- **Heisenberg field algebra**

$$\phi_H(f) := \int d^4x \sum_i \phi_i(x) f_i(x), \qquad f \in \mathcal{S}(\mathbb{R}^4)$$

- **Canonical field algebra**

$$\phi_C(f) := \int d^3x \sum_i \phi_i(t=0,\mathbf{x}) f_i(\mathbf{x}) \qquad f \in \mathcal{S}(\mathbb{R}^3)$$

$$\pi_C(f) := \int d^3x \sum_i \pi_i(t=0,\mathbf{x}) f_i(\mathbf{x}) \qquad f \in \mathcal{S}(\mathbb{R}^3)$$

- **Light-front field algebra**

$$\phi_{LF}(f) := \int \frac{d\tilde{\mathbf{x}}}{2} \sum_i \phi_i(x^+=0,\tilde{\mathbf{x}}) f_i(\tilde{\mathbf{x}}) \qquad \tilde{\mathbf{x}} = (x^-, \mathbf{x}_\perp)$$

$$f \in \mathcal{S}_{ss}(\mathbb{R}^3) \qquad \hat{f}(p^+, \mathbf{p}_\perp)/p^+ \to 0 \qquad \textbf{as} \qquad p^+ \to 0$$

Vacuum functionals

- **Vacuum - linear functional, L, on the field algebra:**

$$|\psi\rangle = A(\phi_x)|0\rangle \qquad \textbf{GNS construction}$$

$$\langle\psi|\chi\rangle = \langle 0|A^\dagger(\phi_x)B(\phi_x)|0\rangle := L(A^\dagger(\phi_x)B(\phi_x))$$

- **Heisenberg vacuum - defined by Wightman distributions**

$$_H\langle 0|\phi_{H1}(f_1)\cdots\phi_{Hn}(f_n)|0\rangle_H$$

- **Canonical vacuum,** $\phi(\mathbf{x},0), \qquad \pi(\mathbf{x},0) \to a(\mathbf{p})$

$$a(\mathbf{p})|0\rangle_C = 0 \qquad {}_C\langle 0|\phi_{C1}(f_1)\cdots\phi_{Cn}(f_n)|0\rangle_C$$

- **Light-front vacuum,** $\phi(0,x^-,\mathbf{x}_\perp,0) \to a(\tilde{\mathbf{p}})$

$$a(\tilde{\mathbf{p}})|0\rangle_{LF} = 0 \qquad \tilde{\mathbf{p}} = (p^+,\mathbf{p}_\perp)$$

$$_{LF}\langle 0|\phi_{LF1}(f_1)\cdots\phi_{LFn}(f_n)|0\rangle_{LF}$$

- $|0\rangle_{LF}$ **does not have enough information to uniquely determine the Heisenberg vacuum,** $|0\rangle_H$**.**

Irreducibility $[O, a_i] = [O, a_i^\dagger] = 0 \, \forall i \;\rightarrow\; O = cI$

- **Canonical case - operators depend on mass (dynamics)**

$$a(\mathbf{p}) = \frac{1}{\sqrt{2\omega_{m_i}(\mathbf{p})}}(\omega_{m_i}(\mathbf{p})\hat{\phi}(\mathbf{p})_{x^0=0} + i\hat{\pi}(-\mathbf{p})_{x^0=0}),$$

$$a^\dagger(\mathbf{p}) = \frac{1}{\sqrt{2\omega_{m_i}(\mathbf{p})}}(\omega_{m_i}(\mathbf{p})\hat{\phi}(\mathbf{p})_{x^0=0} - i\hat{\pi}(-\mathbf{p})_{x^0=0})$$

- **Light-front case - operators kinematic (independent of mass)**

$$a(\tilde{\mathbf{p}}) = \sqrt{2p^+}\theta(p^+)\hat{\phi}(\tilde{\mathbf{p}})_{x^+=0} \qquad a^\dagger(\tilde{\mathbf{p}}) = \sqrt{2p^+}\theta(p^+)\hat{\phi}(-\tilde{\mathbf{p}})_{x^+=0}$$

- **Relation:**

$$a(\tilde{\mathbf{p}}) := a(\mathbf{p})\sqrt{\frac{\omega_m(\mathbf{p})}{p^+}} \qquad |\frac{\partial \tilde{\mathbf{p}}}{\partial \mathbf{p}}| = \frac{p^+}{\omega_m(\mathbf{p})}$$

Free fields

- **Heisenberg fields can be expressed in terms of canonical or light-front creation and annihilation operators.**

$$+$$

- **Creation and annihilation operators can be expressed in terms of canonical equal time fields or fields on the light front.**

$$\Downarrow$$

- **Algebra of Heisenberg fields can be expressed in terms of canonical or light front algebra.**

- **Heisenberg $\to$ canonical**

$$\phi_H(f) = \int d\mathbf{y}(f_\phi(\mathbf{y})\phi_C(\mathbf{y}) + f_\pi(\mathbf{y})\pi_C(\mathbf{y}))$$

where

$$f_\phi(\mathbf{y}) = \int d^4x f(x)\mathcal{K}_{\phi m}(x,\mathbf{y}) \qquad f_\pi(\mathbf{y}) = \int d^4x f(x)\mathcal{K}_{\pi m}(x,\mathbf{y})$$

$$f_\phi(\mathbf{y}), f_\pi(\mathbf{y}) \in \mathcal{S}(\mathbb{R}^3)$$

- **Heisenberg $\to$ light front**

$$\phi_H(f) = \int d\tilde{\mathbf{y}} f_m(\tilde{\mathbf{y}})\phi_{LF}(\tilde{\mathbf{y}})$$

$$f_m(\tilde{\mathbf{y}}) = \int d^4x f(x)\mathcal{K}_m(x,\tilde{\mathbf{y}}) = \mathcal{K}_m(f,\tilde{\mathbf{y}}) \qquad f_m(\tilde{\mathbf{y}}) \in \mathcal{S}_{ss}(\mathbb{R}^3)$$

$$\mathcal{K}_m(x,\tilde{\mathbf{y}}) := \int \frac{d\tilde{\mathbf{p}}}{(2\pi)^3} e^{-ix^+\frac{\mathbf{p}_\perp^2+m^2}{2p^+}} e^{-i\frac{p^+\cdot(x^- - y^-)}{2} + i\mathbf{p}_\perp\cdot(\mathbf{x}_\perp - \mathbf{y}_\perp)}.$$

- **Kernel, $\mathcal{K}_{\phi m}$, transforms covariant generators to light-front generators. For example:**

$$2i\frac{\partial}{\partial x^+}\mathcal{K}_m(x,\tilde{\mathbf{p}}) = \mathcal{K}_m(x,\tilde{\mathbf{p}})\frac{\mathbf{p}_\perp^2 + m^2}{p^+}$$

where

$$\mathcal{K}_m(x,\tilde{\mathbf{p}}) := \int \mathcal{K}_m(x,\tilde{\mathbf{y}})\frac{dy^+ d\mathbf{y}_\perp}{2} e^{i\tilde{\mathbf{y}}\cdot\tilde{\mathbf{p}}}.$$

Interacting fields (assumptions - Wightman axioms, asymptotic completeness)

- **IN fields: irreducible and unitarily equivalent to free fields.**

$$\int \Phi_{IN}(x) f^*(x) d^4x = \int \Phi_{IN}(y^+ = 0, \tilde{\mathbf{y}}) \mathcal{K}_m(x, \tilde{\mathbf{y}}) f^*(x) d^4x =$$

$$\int \Phi_{IN}(y^+ = 0, \tilde{\mathbf{y}}) f_m(\tilde{\mathbf{y}}) \frac{dy^- d\mathbf{y}_\perp}{2}.$$

- **Irreducibility implies Heisenberg fields can be expressed in terms of normal products of IN fields (Glaser, Lehmann, Zimmerman 1957):**

$$\Phi_H(f) =$$

$$\sum \int R_n(x; x_1 \cdots x_n) f(x) d^4x : \Phi_{IN}(x_1) \cdots \Phi_{IN}(x_n) : \prod_k d^4x_k.$$

Combining these results

- **The smeared interacting Heisenberg fields,**

$$\Phi_H(f) =$$

$$\sum \int R_n(x;x_1 \cdots x_n) f(x) d^4x \prod_k d^4x_k d\tilde{\mathbf{y}}_k \mathcal{K}_{m_k}(x_k,\tilde{\mathbf{y}}_k) \times$$

$$:\Phi_{LF}(\tilde{\mathbf{y}}_1) \cdots \Phi_{LF}(\tilde{\mathbf{y}}_N):$$

can be expressed as elements of a complicated dynamical sub algebra of the free-field light-front Fock algebra.

- **Vacuum expectation values depend on the sub algebra but again are independent of the choice of vacuum on the light front.**

Summary - free fields

- **Interchangeable vacuum functionals**

$$ {}_{H1}\langle 0|A|0\rangle_{H1} = {}_{LF1}\langle 0|\mathcal{K}_1 A|0\rangle_{LF1} = {}_{LF2}\langle 0|\mathcal{K}_1 A|0\rangle_{LF2} $$

$$ {}_{H1}\langle 0|A|0\rangle_{H1} \neq {}_{H2}\langle 0|A|0\rangle_{H2} $$

$$ {}_{LF1}\langle 0|\mathcal{K}_1 A|0\rangle_{LF1} \neq {}_{LF1}\langle 0|\mathcal{K}_2 A|0\rangle_{LF1} $$

- **Vacuum expectation values of smeared Heisenberg fields can be expressed as light-front vacuum expectation values of a sub algebra of the light-front field algebra.**

- **The result depends on the sub algebra but is independent of the choice of light-front vacuum.**

The problem of inequivalent representations of the canonical commutation relations, $[q, p] = i$.

- **Stone Von Neumann theorem (2 harmonic oscillators)**

$$q = \frac{1}{\sqrt{2\omega}}\left(a + a^\dagger\right) \qquad p = -i\sqrt{\frac{\omega}{2}}\left(a - a^\dagger\right)$$

$$a' = \cosh(\eta)a + \sinh(\eta)a^\dagger$$

$$\cosh(\eta) := \frac{1}{2}\left(\sqrt{\frac{\omega'}{\omega}} + \sqrt{\frac{\omega}{\omega'}}\right)$$

$$[q', p'] = i$$

- **Canonical transformation $\rightarrow$ unitary transformation**

$$U = e^{iG} \qquad G = (-\frac{i}{2}\eta(a_1 a_1 - a_1^\dagger a_1^\dagger))$$

- **Infinite number of degrees of freedom (QFT)**

$$\|G|\psi\rangle\| = \infty \qquad \forall\, |\psi\rangle, \qquad |0'\rangle = U|0\rangle = \infty$$

Canonical free fields

$$a_2(\mathbf{p}) = \cosh(\eta(\mathbf{p}))a_1(\mathbf{p}) + \sinh(\eta(\mathbf{p}))a_1^\dagger(\mathbf{p})$$

- **where**

$$\cosh(\eta(\mathbf{p})) := \frac{1}{2}\left(\sqrt{\frac{\omega_{m_2}(\mathbf{p})}{\omega_{m_1}(\mathbf{p})}} + \sqrt{\frac{\omega_{m_1}(\mathbf{p})}{\omega_{m_2}(\mathbf{p}}}\right)$$

$$\|G|0\rangle_1\|^2 = \frac{1}{4}\int \eta(\mathbf{p})^2 d\mathbf{p}\delta(0) = \infty.$$

- **Representations inequivalent for** $m_1 \neq m_2$
- **While the light-front field algebras for different masses are unitarily equivalent, the sub algebras associated with the Heisenberg algebras are not.**

Trivial vacuum

$$P^+|0\rangle_f = 0 \qquad P^+ = \sum_i P_i^+; \qquad P_i^+ \geq 0$$

$$V := M - M_0$$

$$P^+ V|0\rangle_f = VP^+|0\rangle_f = 0.$$

$${}_f\langle 0|V^\dagger V|0\rangle_f = \int |\langle p^+, d|V|0\rangle|^2 d\mu(p^+) dd = |{}_f\langle 0|V|0\rangle_f|^2$$

$$V|0\rangle_f = |0\rangle_f {}_f\langle 0|V|0\rangle_f$$

$$0 = M^2|0\rangle_f = (M_0^2 + VM_0 + M_0 V + V^2)|0\rangle_f =$$

$$V^2|0\rangle_f = |0\rangle_f {}_f\langle 0|V|0\rangle_f^2$$

Trivial vacuum

- **The above argument ignores the need for renormalization - in a ϕ^4 interaction the terms with 4 creation operators have singularities that invalidate the above argument:**

$$\int \frac{\theta(p^+)\delta(p^+)dp^+}{(p^+)^2 \prod \xi_i} \prod d\mathbf{p}_{i\perp} d\xi_i \delta(\sum \mathbf{p}_{i\perp})\delta(\sum \xi_i - 1)\times$$

$$a^\dagger(\xi_1 p^+, \mathbf{p}_{\perp 1}) a^\dagger(\xi_2 p^+, \mathbf{p}_{\perp 2}) a^\dagger(\xi_3 p^+, \mathbf{p}_{\perp 3}) a^\dagger(\xi_4 p^+, \mathbf{p}_{\perp 4}).$$

- **Once the theory has been renormalized, any state in the Hilbert space can be expressed by applying an operator from the dynamical light-front sub algebra to the vacuum of any theory.**
- **While the true vacuum is not trivial, it agrees with the trivial vacuum on operators in the dynamical light-front sub algebra.**

The ill-posed initial value problem

- **Heisenberg algebra $\to$ Light front mass m sub algebra**

$$\phi(f) = \int \frac{dy^+ d\mathbf{y}_\perp}{2} f_m(\tilde{\mathbf{y}})\phi(y^+ = 0, \tilde{\mathbf{y}}) := \phi_{LF}(f_m)$$

$$\hat{f}_m(\tilde{\mathbf{p}}) = \sqrt{2\pi}\hat{f}(\frac{\mathbf{p}_\perp^2 + m^2}{p^+}, \tilde{\mathbf{p}}) = \int d^4x f(x) \mathcal{K}_m(x, \tilde{\mathbf{p}})$$

$$\mathcal{K}_m(x, \tilde{\mathbf{p}}) := \int \mathcal{K}_m(x, \tilde{\mathbf{y}}) \frac{dy^+ d\mathbf{y}_\perp}{2} e^{i\tilde{\mathbf{y}}\cdot\tilde{\mathbf{p}}}$$

Dense domain ($\hat{f}(p)$ compact support)

$$\{f(x) | \hat{f}(p) := \int \frac{d^4x}{(2\pi)^2} e^{-p\cdot x} f(x),$$

$$\hat{f}(p) = 0 \qquad \textbf{for} \qquad (p^0)^2, \mathbf{p}^2 > R^2 < \infty\}$$

$$\Downarrow$$

$$|p^+|, |p^-| < 2R.$$

$$\int \phi(x^+, \tilde{\mathbf{x}}) f(x - a^+) d^4x =$$

$$\sum_{n=0}^{\infty} \frac{(-ia^+)^n}{n!} (\frac{\mathbf{p}_\perp^2 + m^2}{2p^+})^n \hat{f}(\frac{\mathbf{p}_\perp^2 + m^2}{p^+}, \tilde{\mathbf{p}})$$

$$|\sum_{n=0}^{\infty} \frac{(-ia^+)^n}{n!} (\frac{\mathbf{p}_\perp^2 + m^2}{2p^+})^n | \leq \sum_{n=0}^{\infty} \frac{(2Ra^+)^n}{n!} = e^{2Ra^+} < \infty.$$

- x^+ **evolution converges on dense set.**

Rotational covariance zero modes

- P^{-} **and the kinematic subgroup form a closed Lie algebra.**
- **Transverse rotations and space reflections are dynamical operators that are not determined by the light-front Hamiltonian and the kinematic subgroup.**
- **Given a dynamical transverse rotation operator, i.e.** J^2**, all of the Poincaré generators are fixed by the kinematic subgroup and the Poincaré commutation relations**

$$P^{-} := P^{+} - 2[J^2,[J^2,P^{+}]] \qquad \textbf{and} \qquad J^1 := -i[J^2,J^3]$$

- **Consistent Interactions in the transverse rotation operator must satisfy linear and non-linear constraints**

$$[J_I^2,[J_0^2,P^1]] = 0$$

and

$$[J_I^2,[J_I^2,J^3]] + [J_0^2,[J_I^2,J^3]] + i[J_I^2,J_0^1] = 0.$$

Rotational covariance zero modes

- **Rotational covariance is equivalent to invariance with respect to change of orientation of the light front (Karmanov). Invariance with respect to change of orientation implies**

$$U_{\hat{\mathbf{z}}}(R,0) = \Omega_{\pm K_{\hat{\mathbf{z}}}} \Omega^{\dagger}_{\pm [R] K_{\hat{\mathbf{z}}} [R^{-1}]} U_0(R).$$

- **Changes of orientation of the light front transform $p^+ = 0$ divergences to ultraviolet divergences.**
- **Renormalization of both kinds of infinities are constrained by rotational covariance and space reflection symmetry.**
- **It is not sufficient to simply renormalize P^-.**
- **Consistent $p^+ = 0$ renormalization (zero modes) is needed make light-front dynamical calculations consistent with covariant calculations.**

Spontaneous Symmetry Breaking

- **Light-front charge commutes with P^+ - it cannot change the vacuum.**
- **Vacuum functionals and fields have no dynamical content on the light-front Fock algebra; the dynamics enters by restricting to a dynamical sub-algebra.**
- **Current operators are operator-valued distributions - the charge operators do not have to exist; however for local fields a signal for spontaneous symmetry breaking is (Coleman)**

$$\lim_{R \to \infty} \langle 0 | [Q_R, \phi(y)] | 0 \rangle \neq 0,$$

where

$$\langle 0 | [Q_R, \phi(y)] | 0 \rangle := \langle 0 | [\int d\mathbf{x} \chi_R(|\mathbf{x}|) j^0(\mathbf{x}, t), \phi(y)] | 0 \rangle$$

Spontaneous Symmetry Breaking

- **Locality cuts off the integral and the commutator can be expressed in terms of the irreducible set of Heisenberg fields which can be expressed in term of elements of the dynamical sub algebra of the light front Fock algebra.**

- **This does not work for the light front-charge because there is no compact region on the light front where outside of that region $(x - y)$ is always space-like for fixed y and $x^+ = 0$.**

Conclusion revisited

- The algebra of fields on a light front is irreducible, but without dynamical content. The dynamical content is in a sub algebra.

- All light-front vacuum functionals agree on this sub algebra. (while the vacuum is not trivial - the trivial vacuum can be used in this sub algebra).

- Inequivalent representations are associated with different sub algebras of the light-front Fock algebra.

Conclusion revisited (continued)

- **Smeared Wightman distributions can be expressed as trivial light-front vacuum expectation values of a dynamical sub-algebra of the light-front Fock algebra.**

- **The light-front Hamiltonian and kinematic subgroup do not determine the transverse rotation generators or space reflection operators. Full rotational covariance and space reflection symmetry require information, not contained in P^-, that relates renormalization of $p^+ = 0$ divergences to ultraviolet divergences.**

- **Light-front charge operators do not couple to a Goldstone boson, but commutators with the fixed-time charge operators that are sensitive to Goldstone bosons can be expressed in terms of the light-front Fock algebra.**

Thanks - organizers!

Relativistic invariance

$$U(\Lambda_2, a_2)U(\Lambda_1, a_1) = U(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2)$$

- **implies that the infinitesimal generators satisfy the commutation relations:**

$$[P^\mu, P^\nu] = 0, \qquad [J^i, P^j] = i\epsilon^{ijk}P^k, \qquad [J^i, J^j] = i\epsilon^{ijk}J^k,$$

$$[J^i, K^j] = i\epsilon^{ijk}K^k, \qquad [K^i, K^j] = -i\epsilon^{ijk}J^k$$

$$[K^i, P^i] = i\delta^{ij}H \qquad [K^i, H] = iP^i.$$

Relativistic invariance

- The relativistic analog of diagonalizing the Hamiltonian is to decompose $U(\Lambda, a)$ into a direct integral of irreducible representations. This is equivalent to simultaneously diagonalizing the mass and spin Casimir operators of the Lie algebra

$$M^2 = (P^0)^2 - \mathbf{P}^2 \qquad \text{and} \qquad \mathbf{S}^2 = W^2/M^2$$

- where W^μ is the Pauli-Lubanski vector

$$W^\mu = (\mathbf{P} \cdot \mathbf{J}, H\mathbf{J} + \mathbf{P} \times \mathbf{K}).$$

- The transformation properties of states in each irreducible subspace is fixed by group theoretical considerations.

Light-front dynamics

- **Light-front generators are different linear combinations of these generators:**

$$P^1, P^2, P^+ = P^0 + P^3, J^3, K^3, \mathbf{E}_\perp := \mathbf{K}_\perp - \hat{\mathbf{z}} \times \mathbf{J}$$

- **The generators**

$$P^- = P^0 - P^3 \neq P_0^-; \qquad \textbf{and} \qquad \mathbf{F}_\perp := \mathbf{K}_\perp + \hat{\mathbf{z}} \times \mathbf{J} \neq \mathbf{F}_{\perp 0} \qquad \textbf{or}$$

$$\mathbf{J}_\perp = \hat{\mathbf{z}} \times \mathbf{J} \neq \hat{\mathbf{z}} \times \mathbf{J}_{0\perp}$$

- **involve interactions. The dynamical generators can be taken as P^-, F^1, F^2 or equivalently P^-, J^1, J^2.**
- **The operators K^3 and $\mathbf{E}_\perp$, which generate light-front preserving boosts, form a closed sub-algebra.**

Generators are constructed using Noether's theorem

$$\mathcal{L}(x)$$

$$\Downarrow$$

Conserved currents

$$\Downarrow$$

$$T^{\mu\nu}(x) = \eta^{\mu\nu}\mathcal{L} - (\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))})\partial_\alpha \phi(x)\eta^{\alpha\nu})$$

$$M^{\mu\alpha\beta} = T^{\mu\beta}x^\alpha - T^{\mu\alpha}x^\beta$$

$$\Downarrow$$

Kinematic Noether charges

$$P^+ = \int_{x^+=0} \frac{d\mathbf{x}_\perp dx^-}{2} T^{++}(x)$$

$$P^i = \int_{x^+=0} \frac{d\mathbf{x}_\perp dx^-}{2} T^{+i}$$

$$E^i = \int_{x^+=0} \frac{d\mathbf{x}_\perp dx^-}{2} T^{++} x^i$$

$$J^3 = \int_{x^+=0} \frac{d\mathbf{x}_\perp dx^-}{2} \left(x^1 T^{+2}(x) - x^2 T^{+1}\right)$$

$$K^3 = \int_{x^+=0} \frac{d\mathbf{x}_\perp dx^-}{2} T^{++}(x) x^-$$

Dynamical Noether charges

$$P^- = \int_{x^+=0} \frac{d\mathbf{x}_\perp dx^-}{2} T^{+-}(x)$$

$$J^1 = \int_{x^+=0} \frac{d\mathbf{x}_\perp dx^-}{4} \left(x^2(T^{++}(x) - T^{+-}) + x^- T^{+2}\right)$$

$$J^2 = -\int_{x^+=0} \frac{d\mathbf{x}_\perp dx^-}{4} \left(x^- T^{+1}(x) + x^1(T^{++} - T^{+-}))\right)$$