Cluster Properties in Poincaré Invariant Quantum Mechanics

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THE UNIVERSITY OF IOWA

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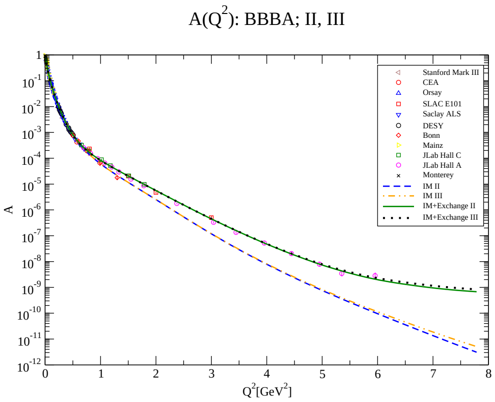
Outline

- Motivation.
- Conventional wisdom.
- Poincaré invariant quantum mechanics.
- Cluster properties (fixed N).
- Solution: fixed N.
- Sokolov-Ekstein operators and Birkhoff lattices.
- Production complications.
- Relation to EFT and RG?

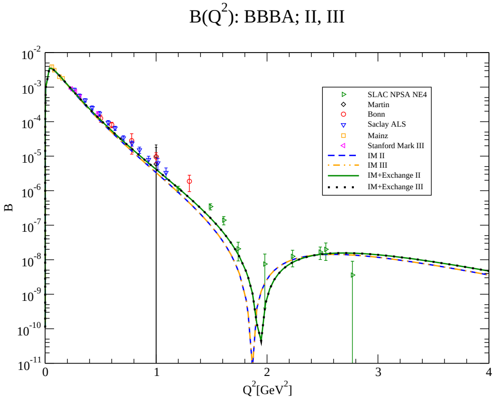
Motivation

- Why use Poincaré invariant quantum mechanics?
  - One of the few methods for treating high-precision calculations at the few GeV scale (see examples).
  - General, consistent with the same axioms as QCD.
  - $\exists$ (many) solution(s) to the inverse scattering problem.
  - Anything less than precise agreement with experiment means that the Hamiltonian is wrong!
  - Predictive power comes from cluster properties.

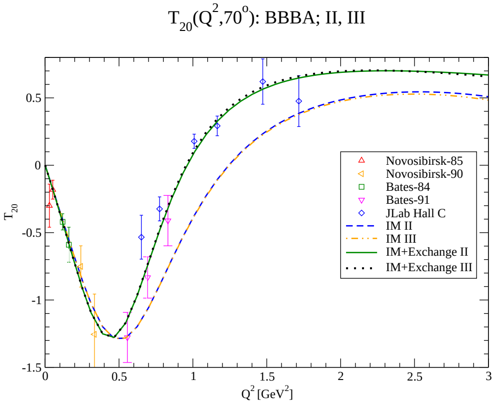
Elastic electron-deuteron scattering A 0-8 GeV^2



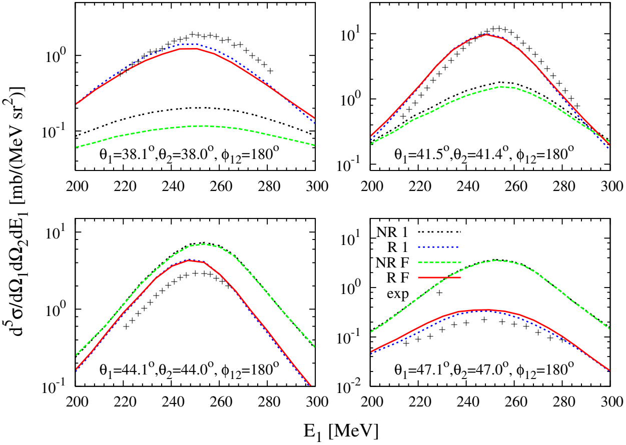
Elastic electron-deuteron scattering B 0-8 GeV^2



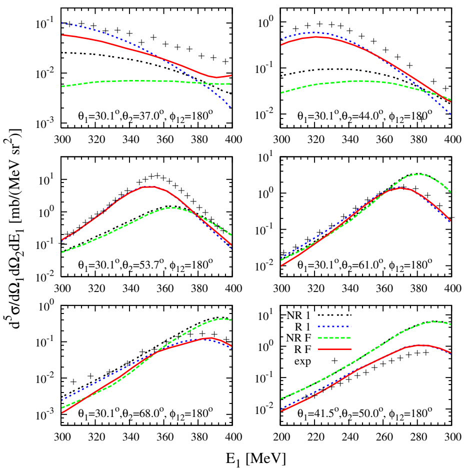
Elastic electron-deuteron scattering T_{20} 0-3 GeV^2



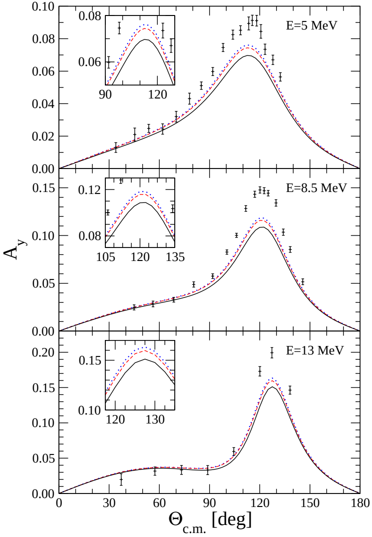
Exclusive Pd breakup at .5 GeV^2



Exclusive Pd breakup at .5 GeV^2



Wigner rotations in A_y at low energies



Conventional Wisdom?

(Weinberg p. 169) : *In relativistic quantum theories, the cluster decomposition principle plays a crucial part in making field theory inevitable. There have been many attempts to formulate a relativistically invariant theory that would not be a local field theory, and it is indeed possible to construct theories that are not field theories and yet yield a Lorentz invariant S-matrix for two-particle scattering, but such efforts have always run into trouble in sectors with more than two particles: either the three particle S-matrix is not Lorentz-invariant, or else it violates the cluster decomposition principle.*

Known counter example (Fixed N)

S. N. Sokolov, Dokl. Akad. Nauk. SSSR 233,575(1977).

How can this be extended to theories with non-trivial particle production?

Is such a solution related to effective field theory?

What is Poincaré invariant quantum mechanics?

- Quantum theory of a bounded number of degrees of freedom.
- The Poincaré group is a symmetry group of the theory.

Poincaré invariant quantum mechanics

(Wigner - 1939)

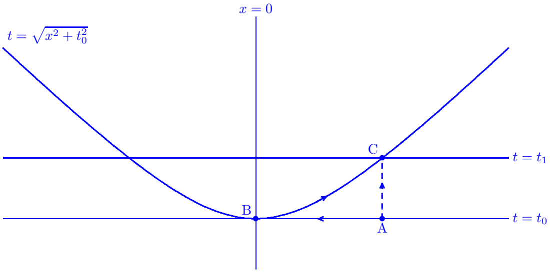
$$P = |\langle \Psi_f | \Psi_i \rangle|^2 = |\langle \Psi'_f | \Psi'_i \rangle|^2 = P'$$

$$\Updownarrow$$

$$|\Psi'\rangle = U(\Lambda, a)|\Psi\rangle$$

Why is Poincaré invariance difficult to implement in a quantum theory?

Consistent initial value problem?



Construction of dynamical $U(\Lambda, a)$ (summary)

- **Identify the relevant particle degrees of freedom.**
- **Choose a basis for single-particle irreps. of the P.G.**
- **Construct a model Hilbert space using direct sums of tensor products of irreducible representation spaces.**
- **Use Poincaré group Clebsch-Gordon coefficients to decompose the tensor products into direct integrals of Poincaré group irreps.**
- **This construction gives non-interacting irreducible representations the Poincaré group.**

$$\Downarrow$$

$$\boxed{U_0(\Lambda, a) : \mathcal{H} \to \mathcal{H} \qquad \text{(Wigner)}}$$

- **Add interactions to the non-interacting mass that commute with the (1) free spin, (2) the operators that label vectors in the non-interacting irreducible subspaces and (3) their canonical conjugates.**

- **Diagonalize M in the non-interacting irreducible basis.**

- **The resulting eigenstates are complete and transform irreducibly with respect to a dynamical representation of the Poincaré group. This defines $U(\Lambda, a)$ on all of $\mathcal{H}$.**

- **The general construction is summarized in the following slides.**

Single-particle basis: $U_1(\Lambda, a)$

$$\Downarrow$$

$$P^\mu, J^{\mu\nu}$$

$$\Downarrow$$

$m, j, \mathbf{h}, \nabla\mathbf{h}$ eg. $\mathbf{h} := (\mathbf{p}, j_z)$ $\nabla\mathbf{h} := (i\nabla_{\mathbf{p}_{|\mathbf{j}}}, \mathbf{j}_\pm)$

$$\mathcal{H}_1 = L^2(\sigma(h))$$

$$\psi(\mathbf{h}) = \langle (m, j)\, \mathbf{h} | \psi \rangle \qquad \sum\!\!\!\!\!\!\int |\psi(\mathbf{h})|^2 d\mathbf{h} < \infty$$

Irreducible representations

$$\langle (m,j)\,\mathbf{h}|U(\Lambda,a)|\psi\rangle = \sum\!\!\!\!\!\!\int \mathcal{D}^{mj}_{\mathbf{h}\mathbf{h}'}(\Lambda,a)d\mathbf{h}'\langle (m,j)\,\mathbf{h}'|\psi\rangle$$

$$\mathcal{D}^{mj}_{\mathbf{h}'\mathbf{h}}(\Lambda,a) = \langle (m,j)\,\mathbf{h}'|U(\Lambda,a)|(m,j)\,\mathbf{h}\rangle$$

$$U(\Lambda,a)|(m,j)\,\mathbf{h}\rangle = \sum\!\!\!\!\!\!\int |(m,j)\,\mathbf{h}'\rangle d\mathbf{h}'\mathcal{D}^{mj}_{\mathbf{h}'\mathbf{h}}(\Lambda,a)$$

The Poincaré group Wigner functions, $\mathcal{D}^{mj}_{\mathbf{h}'\mathbf{h}}(\Lambda,a)$, are known for all j and m.

2-particle Hilbert space

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_1$$

Tensor product basis

$$|(m_1, j_1), \mathbf{h}_1, (m_2, j_2), \mathbf{h}_2\rangle$$

$$\Downarrow$$

$$|\mathbf{h}_1, \mathbf{h}_2\rangle$$

$$\psi(\mathbf{h}_1, \mathbf{h}_2) \qquad \sum\!\!\!\!\!\!\int |\psi(\mathbf{h}_1, \mathbf{h}_2)|^2 d\mathbf{h}_1 d\mathbf{h}_2 < \infty$$

Kinematic representation of the Poincaré group

$$U_0(\Lambda, a) := U_1(\Lambda, a) \otimes U_1(\Lambda, a)$$

$$U_0(\Lambda, a)|(m_1, j_1), \mathbf{h}_1, (m_2, j_2), \mathbf{h}_2\rangle =$$

$$\sum\!\!\!\!\!\!\!\int |(m_1, j_1), \mathbf{h}_1', (m_2, j_2), \mathbf{h}_2'\rangle \prod_{i=1}^{2} d\mathbf{h}_i' \mathcal{D}^{m_i j_i}_{\mathbf{h}_i' \mathbf{h}_i}(\Lambda, a)$$

This is reducible!

Poincaré Clebsch-Gordan coefficients

$$|(m,j),\mathbf{h};\mathbf{d}\rangle = \sum\!\!\!\!\!\!\int |(m_1,j_1),\mathbf{h}_1(m_2,j_2),\mathbf{h}_2\rangle\langle CG\rangle$$

$$\bigotimes_{i=1}^{2}\mathcal{D}^{m_i j_i}_{\mathbf{h}_i'\mathbf{h}_i}(\Lambda,a)\langle CG\rangle = \int_{\oplus m,j,d}\langle CG\rangle\,\mathcal{D}^{mj}_{\mathbf{h}'\mathbf{h}}(\Lambda,a)$$

$$\Downarrow$$

$$\boxed{U_0(\Lambda,a)|(m,j),\mathbf{h};\mathbf{d}\,\rangle = \sum\!\!\!\!\!\!\int |(m,j),\mathbf{h}';\mathbf{d}\,\rangle d\mathbf{h}'\mathcal{D}^{mj}_{\mathbf{h}'\mathbf{h}}(\Lambda,a)}$$

The Poincaré group Clebsch-Gordan coefficients, $\langle CG\rangle$, are known!

3-nucleon Hilbert space

$$\mathcal{H} = \bigotimes_{i=1}^{3} \mathcal{H}_1$$

Tensor product basis

$$\left|(m_1, j_1), \mathbf{h}_1, (m_2, j_2), \mathbf{h}_2, (m_3, j_3), \mathbf{h}_3\right\rangle$$

$$\Downarrow$$

$$\left|\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3\right\rangle$$

Kinematic representation of the Poincaré group

$$U_0(\Lambda, a) := \bigotimes_{i=1}^{3} U_i(\Lambda, a)$$

$$U_0(\Lambda, a)|(m_1, j_1), \mathbf{h}_1, \cdots, (m_3, j_3), \mathbf{h}_3\rangle =$$

$$\sum\!\!\!\!\!\!\!\int |(m_1, j_1), \mathbf{h}'_1, \cdots, (m_3, j_3), \mathbf{h}'_3\rangle \prod_{i=1}^{3} d\mathbf{h}'_i \mathcal{D}^{m_i j_i}_{\mathbf{h}'_i \mathbf{h}_i}(\Lambda, a)$$

This is reducible!

Poincaré Clebsch-Gordan coefficients

$$|(m,j),\mathbf{h};\mathbf{d}\rangle = \sum\!\!\!\!\!\!\int |(m_1,j_1),\mathbf{h}_1,\cdots(m_3,j_3),\mathbf{h}_3\rangle\langle CG\rangle$$

$$\bigotimes_{i=1}^{3}\mathcal{D}^{m_ij_i}_{\mathbf{h}_i'\mathbf{h}_i}(\Lambda,a)\langle CG\rangle = \int_{\oplus m,j,d}\langle CG\rangle\, D^{mj}_{\mathbf{h}'\mathbf{h}}(\Lambda,a)$$

$$\Downarrow$$

$$\boxed{U_0(\Lambda,a)|(m,j),\mathbf{h};\mathbf{d}\,\rangle = \sum\!\!\!\!\!\!\int |(m,j),\mathbf{h}';\mathbf{d}\,\rangle d\mathbf{h}'\mathcal{D}^{mj}_{\mathbf{h}'\mathbf{h}}(\Lambda,a)}$$

The Poincaré group Clebsch-Gordan coefficients, $\langle CG\rangle$, are known!

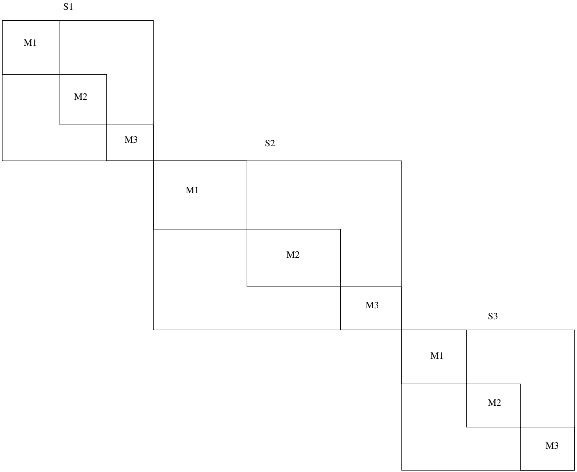
How do we construct a dynamical $U(\Lambda, a)$?

Any unitary representation of the Poincaré group can be decomposed into irreducible representations

$$U(\Lambda, a) = \sum\!\!\!\!\!\!\int \oplus_{m,j} U_{m,j}(\Lambda, a)$$

Mass m spin j irreducible representations transform like mass m spin j particles.

- **Add interactions to mass the are block diagonal in free spin.**



$$N = 2$$

$$M = M_0 + V$$

$$|(m, j), \mathbf{h}, \mathbf{d}\rangle \to |(k, j), \mathbf{p}, \mu, l, s\rangle$$

$$\langle (j, k), \mathbf{p}, \mu, l, s | V | (j', k'), \mathbf{p}', \mu', l', s'\rangle =$$

$$\delta_{jj'} \delta(\mathbf{p} - \mathbf{p}') \delta_{\mu\mu'} \langle k, l, s | v^j | k', l', s'\rangle$$

$$m = M_0(\mathbf{k}^2) = \sqrt{m_1^2 + \mathbf{k}^2} + \sqrt{m_2^2 + \mathbf{k}^2}$$

- **Diagonalize M in non-interacting irreducible basis.**
- **Simultaneous eigenstates of $M, j, \mathbf{p}, j_z$ are complete and transform irreducibly.**

Dynamical $U(\Lambda, a)$

$$\langle (j,k), \mathbf{p}, \mu, l, s, |(j',\lambda), \mathbf{p}', \mu' \rangle = \delta_{jj'}\delta_{\mu\mu'}\delta(\mathbf{p}-\mathbf{p}')\phi_{j,\lambda}(k,l,s)$$

$$(\lambda - \sqrt{m_1^2 + k^2} + \sqrt{m_2^2 + k^2})\phi_{j,\lambda}(k,l,s) =$$
$$\sum_{s',l'} \int_0^\infty \langle k, l, s | v^j | k', l', s' \rangle k^2 dk \phi_{j,\lambda}(k', l', s')$$

$$U(\Lambda, a)|(j,\lambda), \mathbf{p}, \mu\rangle = \sum \int |(j,\lambda), \mathbf{p}', \mu'\rangle d\mathbf{p}' \mathcal{D}^{j\lambda}_{\mathbf{p}',\mu';\mathbf{p},\mu}(\Lambda, a)$$

Structure of M

$$M = M_0 + V_{12} + V_{23} + V_{31}$$

$$V_{ij} = \sqrt{(M_{ij0} + v_{ij})^2 + q^2} - \sqrt{M_{ij0}^2 + q^2}$$

$$\langle k', l', s' | v_{ij}^{j_{ij}} | k, l, s \rangle$$

in three-body irreducible basis

$$|(j, q), \mathbf{p}, \mu, L, S, j_{12}, k_{12}, l, s\rangle$$

This construction can done for systems of any number of particles.

Key property: $\{\mathbf{j}, \mathbf{h}, \nabla \mathbf{h}\} = \{\mathbf{j}_0, \mathbf{h}_0, \nabla \mathbf{h}_0\}$

Interactions involving different particles can be added in this representation and the resulting mass still commutes with $\{\mathbf{j}_0, \mathbf{h}_0, \nabla \mathbf{h}_0\}$

Cluster properties

$\mathbf{A}_i$ = subsystem $\qquad$ $\mathbf{S}$ = system

$$T_{\mathbf{A}_i}(x_i) := U_{\mathbf{A}_i}(I, x_i)$$

$$\mathbf{S} = \mathbf{A}_1 \cup \cdots \cup \mathbf{A}_m$$

$$\mathcal{H} = \mathcal{H}_{\mathbf{A}_1} \otimes \cdots \otimes \mathcal{H}_{\mathbf{A}_m}$$

$$\| [U(\Lambda, a) - \bigotimes_{i=1}^{m} U_{\mathbf{A}_i}(\Lambda, a)] \bigotimes_{j=1}^{m} T_{\mathbf{A}_j}(x_j) |\psi\rangle \| \to 0$$

as $(x_i - x_j)^2 \to +\infty$

Relativistic invariance is valid for isolated sybsystems!

The failure of cluster properties

Irreps and tensor products

$$|(j,m),\mathbf{h},\mathbf{d}\rangle$$

"Poincaré-Jacobi" momenta

$$\mathbf{q}_i = \mathbf{\Lambda}^{-1}(\frac{\mathbf{P}}{M_0})p_i \qquad \mathbf{k}_{ij} = \mathbf{\Lambda}^{-1}(\frac{\mathbf{q}_i + \mathbf{q}_j}{m_{120}})q_i$$

3-particle basis vectors

$$|(j,q),\mathbf{p},\mu,L,S,j_{12},k_{12},l,s\rangle \qquad \textbf{(I)}$$

$$|(j_{12},k_{12}),\mathbf{p}_{12},\mu_{12},l,s\rangle \otimes |(j_2,m_3),\mathbf{p}_3,\mu_3\rangle \qquad \textbf{(T)}$$

2-body interactions in three-body Hilbert space

Free particle irreducible basis (I)

$$\langle (j,q),\mathbf{p},\mu,(L,S,j_p,k,l,s)|v_I|(j',q'),\mathbf{p}',\mu',(L',S',j_p',k',l',s')\rangle =$$

$$\delta(X-X')\langle k,l,s|v^{j_p}|k',l',s'\rangle$$

Tensor product of subsystem bases (T)

$$\langle (j_p,k),\mathbf{p}_{12},\mu_{12},(l,s)|v_T|(j_p',k'),\mathbf{p}_{12}',\mu_{12}',(l',s')\rangle\times$$

$$\langle (j_3,m_3),\mathbf{p}_3,\mu_3|(j_3',m_3'),\mathbf{p}_3',\mu_3'\rangle =$$

$$\delta(Y-Y')\langle k,l,s|v^{j_p}|k',l',s'\rangle$$

$\delta(Y-Y')=\delta(X-X')$ **only when** $k=k'$

$2+1$ Scattering

$$\langle (j,q),\mathbf{p},\mu,(L,S,j_p,k,l,s)|\mathbf{S}_I|(j',q')\mathbf{p}',\mu',(L',S',j'_p,k',l',s')\rangle =$$

$$\delta(X-X')\delta(k-k')\langle l,s|\mathbf{s}^{k,j_p}|l',s'\rangle$$

$$\langle (j_p,k),\mathbf{p}_{12},\mu_{12},(l,s)|\mathbf{S}_T|(j'_p,k'),\mathbf{p}'_{12},\mu'_{12},(l',s')\rangle\times$$

$$\langle (j_3,m_3),\mathbf{p}_3,\mu_3|(j'_3,m'_3),\mathbf{p}'_3,\mu'_3\rangle =$$

$$\delta(Y-Y')\delta(k-k')\langle l,s|\mathbf{s}^{k,j_p}|l',s'\rangle$$

$$\delta(Y-Y')\delta(k-k')=\delta(X-X')\delta(k-k')$$

$$\boxed{\mathbf{S}_I=\mathbf{S}_T}$$

Failure of cluster properties ($2+1$ system)

$$\mathbf{q} := \mathbf{\Lambda}^{-1}(\mathbf{P}/M_0(k))p_3$$

$$\mathbf{p}_3 = \mathbf{q} + \mathbf{P}F(k, \mathbf{q}, \mathbf{P})$$

$$\langle k, \mathbf{P}, \mathbf{q} | v_I | k', \mathbf{P}', \mathbf{q}' \rangle =$$

$$\delta(\mathbf{P} - \mathbf{P}')\delta(\mathbf{q} - \mathbf{q}')\langle k | v | k' \rangle$$

$$\langle \mathbf{P}, \mathbf{q}, k | e^{i\mathbf{p}_3 \cdot \mathbf{a}} v_I e^{-i\mathbf{p}_3 \cdot \mathbf{a}} | \mathbf{P}', \mathbf{q}', k' \rangle =$$

$$\delta(\mathbf{P} - \mathbf{P}')\delta(\mathbf{q} - \mathbf{q}')\underbrace{e^{i\mathbf{P}\cdot\mathbf{a}[F(k,\mathbf{q},\mathbf{P}) - F(k',\mathbf{q},\mathbf{P})]}\langle k | v | k' \rangle}$$

$\rightarrow 0$ **as** $|\mathbf{a}| \rightarrow \infty$ **for** $\vec{P} \neq \vec{0}$**.**

Observations

In frames where P $\neq 0$ the interaction V_I between particles 1 and 2 vanishs as particle 3 is moved away!

In all frames the interaction V_T is independent of the position of the spectator particle. The dynamical representation of the Poincaré group clusters.

Both V_I and V_T give identical on-shell scattering matrices

Additional observations

V_I is block diagonal in the three-body non-interacting spin! Interactions involving different pairs of particles can be added in a manner that preserves Poincaré invariance.

V_T is not block diagonal in the three-body non-interacting spin!. Interactions involving different pairs of particles cannot be added and at the same time preserve Poincaré invariance.

The Fix

Ekstein's Theorem

$$\bar{M} := WMW^{\dagger} \qquad \bar{M}_0 := M_0 \qquad WW^{\dagger} = I$$

$$S = \Omega_+^{\dagger}(M, M_0)\Omega_-(M, M_0) \qquad \bar{S} = \Omega_+^{\dagger}(\bar{M}, M_0)\Omega_-(\bar{M}, M_0)$$

$$\boxed{S = \bar{S} \Leftrightarrow \lim_{t \to \pm\infty} \|(W - I)e^{-iM_0 t}|\psi\rangle\| = 0}$$

$$W = \bar{\Omega}_{\pm}(\bar{M}, M_0)\Omega_{\pm}^{\dagger}(M, M_0) + |\bar{B}\rangle\langle B|$$

$$M - M_0 = V \sim \bar{V} = WMW^{\dagger} - M_0$$

(Ekstein - 1960)

Equivalence of the S matrices implies that the Hamiltonians are related by a unitary transformation and that it is not necessary to transform the free Hamiltonian.

Unitary operators satisfying this asymptotic condition form a group of "scattering equivalences". They relate all solutions of the inverse scattering problem.

While scattering equivalences must preserve S-matrix cluster properties, they do not have to preserve cluster properties of $U(\Lambda, a)$ (established by counter example).

Cluster Properties $(2+1)$

$$\begin{array}{ccc}
U_{(12)_0}(\Lambda, a) \otimes U_3(\Lambda, a) & \xrightarrow{\langle AB|C\rangle_0} & U_{(12)_0(3)}(\Lambda, a) \\
V_{(12)(3)T} \downarrow & & \downarrow \bar{V}_{(12)(3)I} \\
U_{(12)_i}(\Lambda, a) \otimes U_3(\Lambda, a) & \xrightarrow{\langle AB|C\rangle_I} & U_{(12)_i(3)}(\Lambda, a) \underbrace{\sim}_{W_{12}} \bar{U}_{(12)_i(3)}(\Lambda, a)
\end{array}$$

$$S_{(12)(3)} = \bar{S}_{(12)(3)} \underbrace{\Rightarrow}_{\textbf{Ekstein}} \bar{U}_{(12)_i(3)}(\Lambda, a) = W_{12}^{\dagger} U_{(12)_i(3)}(\Lambda, a) W_{12}$$

Algebraic cluster properties - turn off interaction → tensor products of subsystem representations

$$M_I = (M_0 + V_{12I}) + (M_0 + V_{23I}) + (M_0 + V_{31I}) - 2M_0 + V_{(123)}$$

$$= M_{12I} + M_{23I} + M_{31I} - 2M_0 + V_{(123)}$$

$$= W_{12}^{\dagger} M_{12T} W_{12} + W_{23}^{\dagger} M_{32T} W_{23} + W_{31}^{\dagger} M_{31T} W_{31} - 2M_0$$

$$+ V_{(123)}$$

$$M_T = \prod W_{ij} M_I \prod W_{kl}$$

W_{ij} scattering equivalence $\to \prod W_{ij}$ is a scattering equivalence

Turn off interactions involving particle $k \to$

$$U_T(\Lambda, a) \to U_{ij}(\Lambda, a) \otimes U_k(\Lambda, a)$$

M_T has three-body forces and is not block diagonal in the free spin

W_{ij} do not commute. Sokolov showed how to construct unitary symmetrized products, $W = \exp(\sum \ln W_{ij})$

N-Particles

$$M = -W^{\dagger}[\sum_{a \neq 1} \delta^{-1}_{1 \supset a} W_a M_a W_a^{\dagger}] W$$

Birkhoff lattices with respect to partial ordering are needed to recursively construct W_a

$$\delta^{-1}_{a \supset b} = (-)^{n_a} \prod_{i=1}^{n_a} (-)^{n_{b_i}} (n_{b_i} - 1)! \qquad a \supset b$$

$$\delta_{a \cap b \supset b} = \delta_{a \supset c} \delta_{b \supset c} \qquad \delta_{a \supset b \cup c} = \delta_{a \supset b} \delta_{a \supset c}$$

$$U_a(\Lambda, a) = \bigotimes_{i=1}^{n_{a_i}} U_{a_i}(\Lambda, a)$$

Key elements of general construction

- **The construction starts with a few-body problem. The solution of the few-body problem is constrained by experiment.**
- **There is quantity, N, that can be used to paramaterize an inductive construction.**
- **There is a partial ordering of refinements for cluster properties, $(a \subseteq b) \to (1)(2)(23) \subseteq (1)(234)$.**
- **N-body Hilbert can be decomposed into a tensor product of subsystem Hilbert spaces, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_{23}$ that are related to the partial ordering.**
- **Few-body interactions in the N-particle Hilbert space: Scattering equivalence, (W_a), between the tensor product and irreducible form of the interaction.**

Production questions?

- Is there a few-body problem that can be compared to experiment? (finite number of degrees of freedom and can be directly compared to experiment).

    - Choose the relevant degrees of freedom to be physical particles with CM energy limited to an experimentally relevant scale.

    - No bare particles, no elementary production vertices.

    - The Hilbert space is the direct sum of tensor products of irreducible representation spaces associated with physical particles.

Production questions?

- Is there an ordering on complexity that replaces N in the fixed N case?
  - Identify and order all physical particle production thresholds. Physics between successive thresholds replaces physics for a given particle number.
  - Because the thresholds are related to eigenvalues of the mass Casimir operator, Poincaré CG coefficients can be used to decompose the Hilbert space into an orthogonal direct sum of irreducible subspaces.

$$\mathcal{H} = \bigoplus_N \mathcal{H}_N \qquad \rightarrow \qquad \mathcal{H} = \bigoplus_a \mathcal{H}_{\Delta_a}$$

Production questions?

- **Is there a partial ordering and associated Birkhoff lattice structure that respects cluster properties?**

  - $a_1 a_2 \cdots a_n \subseteq a$ **if and only if** $\sum_i m_{i\,max} \leq m_{a\,max}$

$$\mathcal{H}_{\Delta_a} = \Pi \left(\otimes \mathcal{H}_{\Delta_{a_i}} \right) \oplus \mathcal{H}_r$$

Production questions?

- **What is the appropriate formulation of the cluster property?**

$$\lim_{(x_i - x_j)^2 \to \infty} \| \left(U(\Lambda, a) - \bigotimes_i U_{a_i}(\Lambda, a) \right) \bigotimes_i U_{a_i}(I, x_i) \Pi |\Psi\rangle \| = 0$$

Production questions?

- **Is there a dynamical model with operators $U_{\Delta_a} : \mathcal{H}_{\Delta_a} \to \mathcal{H}_{\Delta_a}$?**
  - **Yes - the interactions must be block diagonal in mass interval Δ_a.**
  - **It extend to $U_0(\Lambda, a)$ on the orthogonal complement of $\mathcal{H}_{\Delta_a}$.**
- **Is there a subsystem tensor product and scattering equivalent irreducible dynamical model?**
  - **Yes - the block diagonal nature of the interactions and the extension the orthogonal complement of $\mathcal{H}_{\Delta_a}$ are needed.**

$$\langle (k,j),\mathbf{p},\mu,\mathbf{d}|\langle(\lambda,j'),\mathbf{p}',\mu'\rangle = \delta(\mathbf{p}-\mathbf{p}')\delta_{\mu\mu'}\delta_{jj'}\phi(k,d)$$

$$(\lambda - M_0(k))\phi(k,d) = \sum \int_0^{\Delta_i} \langle k,d|v|k',d\rangle k'^2 dk' \phi(k',d')$$

Conclusions - speculations?

- **It is possible to construct exactly Poincaré invariant quantum models for fixed numbers of particles.**

- **These models can be used in few-body calculations at the few GeV scale, however physics at this scale requires a consistent treatment of production.**

- **The elements used in the fixed N construction generalize, but cluster properties are exactly realized only up to a scale.**

- **The dynamical models must be built out of (effective) models that are valid in a limited energy range.**

Thanks! - The INT and workshop organizers

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