Osterwalder-Schrader Stability

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Quasi-Schwinger Functions

$$S_n(\vec{\mathsf{x}}_1, \mathsf{x}_1^0, \cdots, \vec{\mathsf{x}}_n, \mathsf{x}_n^0) =$$

$$= \lim_{\phi \to \frac{\pi}{2}} G_n(\vec{\mathsf{x}}_1, e^{i\phi}\mathsf{x}_1^0, \cdots, \vec{\mathsf{x}}_n, e^{i\phi}\mathsf{x}_n^0)$$

$$G_n \Rightarrow S_n$$

Quasi-Schwinger Functions

- Analyticity of G_n in times follows from spectral properties:

$$\hat{f}(t) = \int_0^\infty e^{iEt} f(E) dE$$

$$\text{supp}(f) \in [0, \infty)$$

$$\hat{f}(t) \to \hat{f}(t + i\tau) \qquad \tau > 0$$

$$\hat{f}(t) \to \hat{f}(e^{i\phi}t) \qquad t > 0, \quad 0 \leq \phi \leq \frac{\pi}{2}$$

Quasi-Wightman Functions

- quasi-Wightman functions can be recovered from quasi-Schwinger functions as boundary values of the analytic functions

$$W_n(\vec{\mathrm{x}}_1, x_1^0, \cdots, \vec{\mathrm{x}}_N, x_N^0) =$$

$$\lim_{\mathrm{x}_1^0 > \mathrm{x}_2^0 > \cdots > \mathrm{x}_N^0 \to 0} S_n(\vec{\mathrm{x}}_1, \mathrm{x}_1^0 + ix_1^0, \cdots, \vec{\mathrm{x}}_N, \mathrm{x}_N^0 + ix_N^0)$$

- Order of Euclidean times $=$ order of fields in W

$$S_n \Rightarrow W_n$$

Physical Hilbert Space

$$\langle f|g\rangle = (\Theta f, Sg) = (f, \Theta S g) =$$

$$\sum_{m,n} \int d^4\mathsf{x}_1 \cdots d^4\mathsf{x}_{m+n} f_n^*(\theta \mathsf{x}_n, \cdots, \theta \mathsf{x}_1) \times$$

$$S_{m+n}(\mathsf{x}_1, \cdots, \mathsf{x}_{m+n}) g_m(\mathsf{x}_{n+1}, \cdots, \mathsf{x}_{n+m}).$$

Reflection Positivity

$$\text{supp}\langle \mathsf{x}|f\rangle : \mathsf{t}_{k1} > \mathsf{t}_{k2} \cdots > \mathsf{t}_{kk} > 0 \quad (\langle \mathsf{x}|f\rangle \in \mathcal{S}_{>})$$

$$(\Theta f, Sg) \geq 0$$

Poincaré Group

$$\langle \mathsf{x}|\vec{P}|f\rangle :=$$

$$\{0, -i\frac{\partial}{\partial \vec{\mathsf{x}_{11}}} f_1(\mathsf{x}_{11}), (-i\frac{\partial}{\partial \vec{\mathsf{x}_{21}}} - i\frac{\partial}{\partial \vec{\mathsf{x}_{22}}}) f_2(\mathsf{x}_{21}, \mathsf{x}_{22}), \cdots\}$$

$$\langle \mathsf{x}|\vec{J}|f\rangle := \{0, -i\vec{\mathsf{x}}_{11} \times \frac{\partial}{\partial \vec{\mathsf{x}}_{11}} f_1(\mathsf{x}_{11}),$$

$$(-i\vec{\mathsf{x}}_{21} \times \frac{\partial}{\partial \vec{\mathsf{x}}_{21}} - i\vec{\mathsf{x}}_{22} \times \frac{\partial}{\partial \vec{\mathsf{x}}_{22}}) f_2(\mathsf{x}_{21}, \mathsf{x}_{22}), \cdots\}$$

Poincaré Group

$$\langle \mathsf{x}|H|f\rangle :=$$

$$\{0, \frac{\partial}{\partial \mathsf{x}_{11}^0} f_1(\mathsf{x}_{11}), \frac{\partial}{\partial \mathsf{x}_{21}^0} + \frac{\partial}{\partial \mathsf{x}_{22}^0} f_2(\mathsf{x}_{21}, \mathsf{x}_{22}), \cdots\}$$

$$\langle \mathsf{x}|\vec{B}|f\rangle := \{0, \vec{\mathsf{x}}_{11} \frac{\partial}{\partial \mathsf{x}_{11}^0} - \mathsf{x}_{11}^0 \frac{\partial}{\partial \vec{\mathsf{x}}_{11}} f_1(\mathsf{x}_{11}),$$

$$(\vec{\mathsf{x}}_{21} \frac{\partial}{\partial \mathsf{x}_{21}^0} - \mathsf{x}_{21}^0 \frac{\partial}{\partial \vec{\mathsf{x}}_{21}} + \vec{\mathsf{x}}_{22} \frac{\partial}{\partial \mathsf{x}_{22}^0} - \mathsf{x}_{22}^0 \frac{\partial}{\partial \vec{\mathsf{x}}_{22}}) f_2(\mathsf{x}_{21}, \mathsf{x}_{22}), \cdots\}.$$

Dynamics - Euclidean RQM

- H, $\vec{P}$, $\vec{J}$, $\vec{B}$ well defined and self-adjoint on the Physical Hilbert space
- H, $\vec{P}$, $\vec{J}$, $\vec{B}$ satisfy Poincaré commutation relations
- $\langle \cdot | \cdot \rangle$, H, $\vec{P}$, $\vec{J}$, $\vec{B}$ defines a relativistic quantum theory.

Euclidean BS

$$G_4 = G_0 + G_0 K G_4$$

$$\Downarrow$$

$$S_4 = S_0 + S_0 K_e S_4$$

$$G_4 \Leftrightarrow S_4 \Leftrightarrow W_4$$

OS Stability

$$S_4 = S_0 + S_0 K_e S_4$$

$$\Pi_> : S \to S_>$$

$$(f, \Pi_> \Theta S_0 \Pi_> f) \geq 0$$

$K_e$small, Euclidean covariant

$$?$$

$$(f, \Pi_> \Theta S_4 \Pi_> f) \geq 0$$

Theorems

Theorem 1: $(f, \Pi_> \Theta S_0 \Pi_> f) = 0$ for some $f \in \mathcal{S}_>$ implies the BS equation is OS unstable in a neighborhood of $K = 0$.

Theorems

Theorem 2: If $S_0 = S_{10}S_{20}$ with S_{i0} a free particle Schwinger functions, then there are non-zero $f \in \mathcal{S}_>$ such that $(f, \Pi_> \Theta S_0 \Pi_> f) = 0$.

Theorems

Theorem 3: If $S_0 = S_{10}S_{20}$ and S_{i0} has a Lehmann with non-empty continuous spectrum then there are no non-zero $f \in \mathcal{S}_>$ such that $(f, \Pi_> \Theta S_0 \Pi_> f) = 0$.

Null Spaces

$$(f, \Theta S_0 f) =$$

$$\int \left| \int d\mathsf{t} \tilde{f}(\mathsf{p}, \mathsf{t}) \frac{2\pi e^{-\omega_m(\vec{\mathsf{p}})\mathsf{t}}}{\sqrt{\omega_m(\vec{\mathsf{p}})}} \right|^2 d^3\mathsf{p}\, dm\, \rho(m)$$

$$\tilde{f}(\mathsf{p}, \mathsf{t}) = \frac{1}{(2\pi)^{3/2}} \int e^{i\vec{\mathsf{p}}\cdot\vec{\mathsf{x}}} f(\vec{\mathsf{x}}, \mathsf{t}) d^3\mathsf{x}$$

Null Spaces

$$(f, \Theta S_0 f) = 0$$

$$\Downarrow$$

$$\int d\mathsf{t}\, \tilde{f}(\mathsf{p}, \mathsf{t}) e^{-\omega_m(\vec{\mathsf{p}})\mathsf{t}} = 0$$

$$\Downarrow$$

$\forall \vec{\mathsf{p}}$ and all $m \in \mathrm{supp}(\rho)$.

Null Spaces

- S_0 free $\Rightarrow m$ is a fixed number

$$\tilde{f}(\vec{\mathsf{p}},\mathsf{t}) = \tilde{g}(\vec{\mathsf{p}})\xi(\mathsf{t}) \qquad \int_a^b \xi(\mathsf{t})d\mathsf{t} = 1 \qquad \text{supp}(\xi) \in [a, b$$

$$\hat{\xi}(\vec{\mathsf{p}}) := \int_a^b e^{-\omega_m(\mathsf{p})t}\xi(\mathsf{t})d\mathsf{t}$$

$$\tilde{f}(\vec{\mathsf{p}},\mathsf{t}) = \tilde{g}(\vec{\mathsf{p}})\xi(\mathsf{t})[1 - e^{\omega_m(\mathsf{p})t}\hat{\xi}(\vec{\mathsf{p}})]$$

$$\Downarrow$$

$$(f, \Theta S_0 f) = 0$$

Null Spaces

- If $\rho(m)$ has absolutely continuous spectrum then the instability proof breaks down:

$$\int \tilde{f}(\vec{\mathsf{p}}, \mathsf{t}) e^{-\omega_m(\vec{\mathsf{p}})\mathsf{t}} = 0.$$

$$\Downarrow$$

$$\tilde{f}(\vec{\mathsf{p}}, \mathsf{t}) = 0$$

Conclusion

- Euclidean BS with free S_0 is OS unstable at $K = 0$.
- The BS equation with a realistic S_0 may be OS stable for a suitably restricted class of Kernels.
- OS positivity is essential for a quantum mechanical interpretation.

Future

- Models with S_0 having absolutely continuous Lehmann spectra ?
- Case of Fermions (protons, neutrons, quarks)?