Relativistic Quantum Mechanics

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Motivation

- Scattering experiments at the Thomas Jefferson National Laboratory use high-current electron beams to deduce the structure of baryons, mesons, and light nuclei.

$$c\Delta p \Delta x \geq \hbar c \sim .197(GeV \cdot fm)$$

$$\Delta x = .2fm \rightarrow c\Delta p \geq 1GeV \approx m_p c^2$$

- Relativistic quantum mechanics is needed to interpret these experiments!

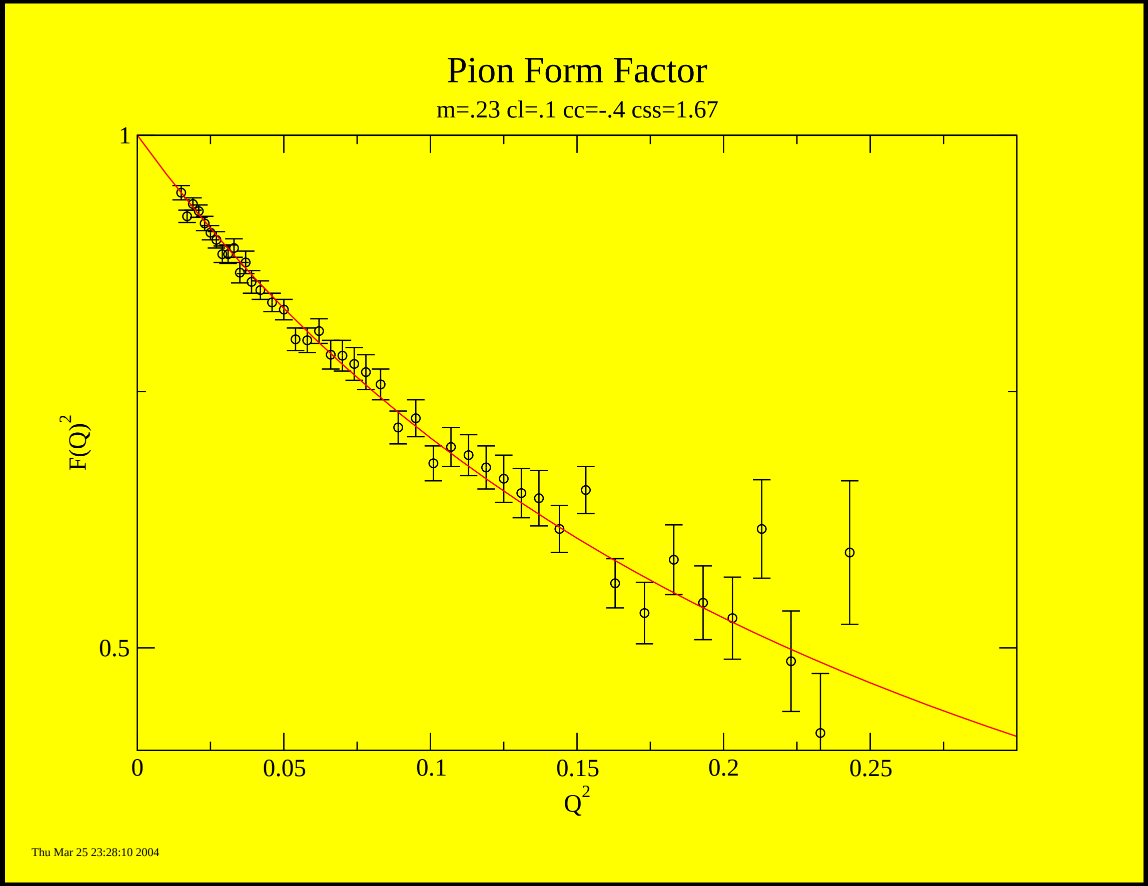
Example

- Assume a model quark-antiquark Hamiltonian:

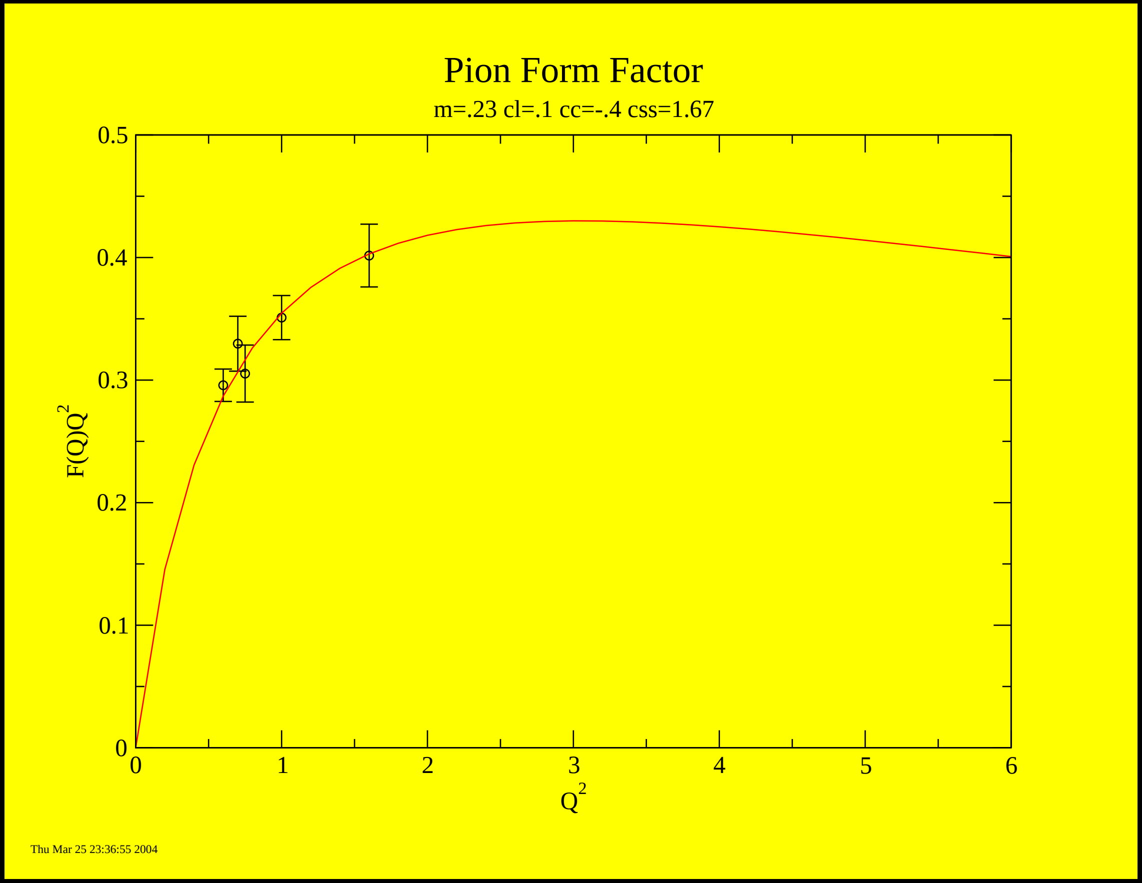
$$M = 2\sqrt{m^2 + p^2} + \alpha r - \frac{\beta}{r} + \gamma \vec{s}_1 \cdot \vec{s}_2 e^{-\lambda r^2}$$

- Choose model parameters to fit experimental data.
- Use the resulting model to make predictions.

Pion form factor



Pion form factor



Pion form factor

- This example makes a prediction, based on a relativistic model, for proposed measurements.
- The nominal RMS radius of the pion in this model is 3.88 fm, which is slightly more than half of the charge radius. This large difference is a relativistic effect.

Quantum Theory

- States are represented by unit vectors, $|\psi\rangle$, in a complex vector space

$$\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$$

- The predictions of a quantum theory are the probabilities

$$P_{\psi\phi} = |\langle\psi|\phi\rangle|^2$$

Relativity

- Inertial coordinate systems are coordinate systems where free particles move with constant velocity.
- The probabilities measured in equivalent experiments done in different inertial coordinate systems are identical.

Relativity

- Because solutions of the Schrödinger equation are not observable in quantum theory, relativity does not require that the Schrödinger equation have the same form in every inertial coordinate system.
- This is an important difference between the classical and quantum formulation of relativistic invariance.

Relativistic Invariance

- Galilean Relativity: X and X' inertial $\Rightarrow$

$$|\Delta\vec{x}_{ij}| = |\Delta\vec{x}_{ij}\,'|$$

$$\Delta t_{ij} = \Delta t'_{ij}$$

- Special Relativity: X and X' inertial $\Rightarrow$

$$|\Delta\vec{x}_{ij}|^2 - c^2|\Delta t_{ij}|^2 = |\Delta\vec{x}'_{ij}|^2 - c^2|\Delta t'_{ij}|^2$$

Relativity

- The Michelson-Morley experiment verified that special relativity gives the observed relation between inertial coordinate systems.
- The most general transformation relating two inertial coordinate systems is a Poincaré transformation:

$$x^\mu \longrightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$$

$$g^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta g^{\alpha\beta}$$

Relativity in Quantum Theory

$$X \to X'$$

$$|\phi\rangle, |\psi\rangle \longrightarrow |\phi'\rangle, |\psi'\rangle$$

$$\Downarrow$$

$$|\langle\phi|\psi\rangle|^2 = |\langle\phi'|\psi'\rangle|^2$$

- Must hold for all $|\psi\rangle$, $|\phi\rangle$ and all inertial coordinate systems X and X'

Wigner's Theorem

$$|\langle\phi|\psi\rangle|^2 = |\langle\phi'|\psi'\rangle|^2$$

$$\Downarrow$$

$$|\phi'\rangle = U|\phi\rangle \qquad |\psi'\rangle = U|\psi\rangle \qquad \langle\psi'|\phi'\rangle = \langle\psi|\phi\rangle$$

or

$$|\phi'\rangle = A|\phi\rangle \qquad |\psi'\rangle = A|\psi\rangle \qquad \langle\psi'|\phi'\rangle = \langle\psi|\phi\rangle^*$$

Wigner's Theorem

- For rotations, translations, and rotationless Lorentz transformations:

$$U = U^{1/2}U^{1/2} \qquad A = A^{1/2}A^{1/2}$$

- The correspondence between states must be unitary!

$$\Downarrow$$

$$|\psi'\rangle = U|\psi\rangle$$

Wigner's Theorem

$$\underbrace{X \underbrace{\rightarrow}_{(\Lambda, a)} X' \underbrace{\rightarrow}_{(\Lambda', a')} X''}_{(\Lambda'', a'')}$$

$$\Downarrow$$

$$U(\Lambda', a')U(\Lambda, a) = U(\Lambda'\Lambda, \Lambda' a + a')$$

- $U(\Lambda, a)$ is a unitary representation of the Poincaré group

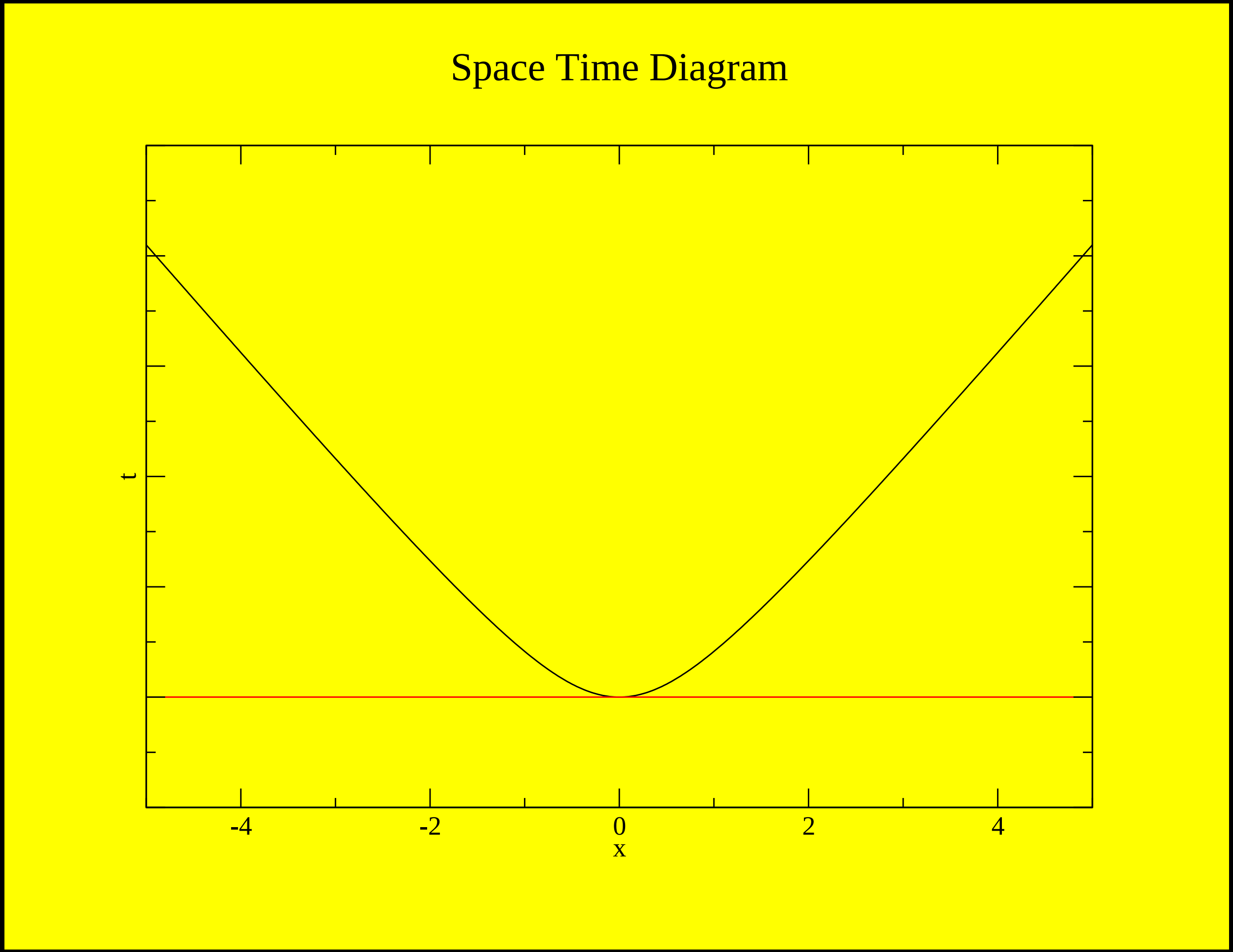
Elements of Relativistic Quantum Theory

- Complex vector space, $\mathcal{V}$ (Hilbert space)
- Unitary representation of the Poincaré group

$$U(\Lambda, a) : \mathcal{V} \to \mathcal{V}$$

- $U(\Lambda, a)$ contains the dynamics (time evolution).

Dynamics



Dynamics

$$\Lambda = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix}$$

$$\vec{a} = (0, 0, 0, z) \quad \vec{b} = (0, 0, 0, -\gamma z) \quad t := (\gamma\beta z, \vec{0})$$

$$(\Lambda^{-1}, \vec{b})(\Lambda, \vec{a}) = (I, (t, \vec{0}))$$

Dynamics

$$U(\Lambda, a) = e^{-i \sum \lambda_i G_i}$$

$$G_i = G_i^\dagger \qquad G_i \in \{H, \vec{P}, \vec{J}, \vec{K}\}$$

$$[G_i, G_j] = i g_{ijk} G_k$$

$$\Downarrow$$

$$[P_i, K_j] = i\delta_{ij} H$$

Dynamics

- Galilean Invariant Dynamics $\{H, \vec{P}, \vec{J}, \vec{K}, M\}$

$$[P_i, K_j] = i\delta_{ij} M$$

- Poincaré Invariant Dynamics $\{H, \vec{P}, \vec{J}, \vec{K}\}$

$$[P_i, K_j] = i\delta_{ij} (H_0 + V)$$

- Puts non-linear constraints on $\{H, \vec{P}, \vec{J}, \vec{K}\}$ for consistent initial value problem!

One-Particle Relativistic Quantum Mechanics

Problem

- Find the single particle Hilbert space $\mathcal{V}$
- Find $U(\Lambda, a)$ on $\mathcal{V}$

One-Particle Relativistic Quantum Mechanics

$$|\langle\Psi|\Phi\rangle| = |\langle\Psi'|\Phi'\rangle|$$

$$\Downarrow$$

$$U(\Lambda, a)$$

$$\Downarrow$$

$$\{H, \vec{P}, \vec{J}, \vec{K}\}$$

$$\Downarrow$$

One-Particle Relativistic Quantum Mechanics

- Construct a maximal set of commuting Hermitian operators out of $\{H, \vec{P}, \vec{J}, \vec{K}\}$:
- Use the commutations relations of $\{H, \vec{P}, \vec{J}, \vec{K}\}$ to determine the eigenvalues of these operators.
- Let $\mathcal{V}$ be the space of square integrable functions of the eigenvalues.
- Use the transformation properties of the observables to determine $U(\Lambda, a)$.

Commuting Observables

$$\{M, \vec{j}^{\,2}, \vec{P}, \vec{j}_z\}$$

where

$$M := \sqrt{H^2 - \vec{P}\cdot\vec{P}}$$

$$\vec{j} = \vec{J} + \frac{1}{2}\{\frac{1}{H}, \vec{K}\} \times \vec{P} + \frac{(\vec{P} \times (H\vec{J} - \vec{P} \times \vec{K})) \times \vec{P}}{MH(M+H)}$$

Commuting Observables

- The mass and spin of the particle fix the eigenvalue of m and j^2
- The commutation relations imply

$$[j_x, j_y] = ij_z \cdots$$

- which means j must be integer of half integer and

$$\vec{j}_z \in \{-j, -j+1, \cdots j-1, j\}$$

One-Particle Hilbert Space

- Lorentz transformations can generate any momentum $\Rightarrow \quad -\infty < P^i < \infty$

$$\Psi(\vec{p}, \mu) = \langle m, j; \vec{p}, \mu | \Psi \rangle$$

$$\langle \Psi | \Phi \rangle := \sum_{\mu=-j}^{j} \int d^3p \Psi^*(\vec{p}, \mu) \Phi(\vec{p}, \mu)$$

$$\langle \Psi | \Psi \rangle < \infty$$

Basis vectors

- Start with eigenstate with $\vec{P} = \vec{0}$ and $j_z = j$:

$$|(m, j)\vec{0}, j\rangle := |\vec{0}, j\rangle$$

- Define the lowering operator $j_- = j_x - ij_y$

$$|\vec{0}, \mu - 1\rangle = \frac{1}{\sqrt{(j + \mu)(j - \mu + 1)}} j_- |\vec{0}, \mu\rangle$$

- This defines

$$|m, j; \vec{0}, \mu\rangle := |\vec{0}, \mu\rangle$$

Basis vectors

- The commutation relations $[j_x, j_y] = ij_z \cdots$ and definitions imply for any rotation R

$$U(R,0)|\vec{0},\mu\rangle = \sum_{\mu'=-j}^{j} |\vec{0},\mu'\rangle D^j_{\mu'\mu}(R)$$

Basis vectors

- Let

$$\Lambda(\vec{p}) : (m, \vec{0}) = (\sqrt{\vec{p}^2 + m^2}, \vec{p}) :$$

$$|\vec{p}, \mu\rangle := U(\Lambda(\vec{p}), 0)|\vec{0}, \mu\rangle(1 + \frac{\vec{p}^2}{m^2})^{-1/4}$$

- the factor $(1 + \frac{\vec{p}^2}{m^2})^{-1/4}$ is determined by the requirement that $U(\Lambda, 0)$ be unitary.

$$U(I, a)|\vec{0}, \mu\rangle := e^{-ia^0 m}|\vec{0}, \mu\rangle$$

One-Body Problem

- Any Poincare transformation can be decomposed into a product of the previously defined transformations

$$U(\Lambda, a) = U(\Lambda(p'), 0)U(I, \Lambda^{-1}(p')a)\times$$

$$U(\underbrace{\Lambda^{-1}(p')\Lambda\Lambda(p)}_{\text{rotation}}, 0)U(\Lambda^{-1}(p), 0)$$

$$p' = \Lambda p$$

The two-body problem

- The one-body solution is well known.
- It provides clues on how to solve the two-body problem

The Currie Jordan Sudarshan Theorem

$$\{X_i, P_j\} = \delta_{ij}$$

$$\{X_i, J_j\} = \epsilon_{ijk} X_k$$

$$\{X_i, K_j\} = X_j \{X_i, H\}$$ world-line condition

- Poisson brackets of $\{H, \vec{P}, \vec{J}, \vec{K}\}$ satisfy Lie algebra of Poincaré group.

The Currie-Jordan-Sudarshan Theorem

$$\Downarrow$$

- Can only be satisfied for free particles!
- The CJS theorem suggests that it might be difficult to formulate relativistic models for systems of interacting particles.
- The out in relativistic quantum mechanics is that there is no position operator!

Position

- Let $|\vec{0}, 0\rangle$ be a vector corresponding the a particle at the origin at time $t = 0$

$$\langle \vec{p}\,|\vec{0}; 0\rangle = \langle \vec{p} = 0|U^\dagger(\Lambda(\vec{p}), 0)U(\Lambda(\vec{p}), 0)|\vec{0}; 0\rangle$$

$$= \frac{N}{(\vec{p}^{\,2} + m^2)^{1/4}}$$

$$\langle \vec{p}\,|\vec{x}; t = 0\rangle = \langle \vec{p}\,|U(I, \vec{x}\,)|\vec{0}; 0\rangle$$

$$\Downarrow$$

Position

$$\langle \vec{p} \,|\, \vec{x}\, \rangle = \frac{e^{-i\vec{p}\cdot\vec{x}}}{(2\pi)^{3/2}(\vec{p}^{\,2} + m^2)^{1/4}}$$

$$\langle \vec{0} | \vec{x} \rangle = \int \frac{d^3p}{(2\pi)^3(\vec{p}^{\,2} + m^2)^{1/2}} e^{-i\vec{p}\cdot\vec{x}} =$$

$$\frac{ic^2}{2}(2\pi)^2 D_+(0, |\vec{x}|) \sim \left(\frac{mc|\vec{x}|}{\hbar}\right) e^{\frac{-mc|\vec{x}|}{\hbar}} \neq 0$$

The Two-Body Problem

- The one-body solution has clues about how to solve the two-body problem
- Find $M, j^2, \vec{P}, j_z$ for the non-interacting two-body system.
- Construct basis of simultaneous eigenstates of these operators

The Two-Body Problem

- Add interactions to M that commute with and do not depend on $\vec{P}$ and j_z
- Construct simultaneous eigenstates of $M + V, j^2, \vec{P}, j_z$
- Construct $U(\Lambda, a)$ following the one body construction

The Two-Body Problem

- Construct

$$\langle \vec{p}_1, \mu_1; \vec{p}_1, \mu_1 | m_{12}, j_{12}; \vec{p}_{12}, \mu_{12}; l_{12}, s_{12} \rangle$$

$$\Downarrow$$

$$| m_{12}, j_{12}; \vec{p}_{12}, \mu_{12}; l_{12}, s_{12} \rangle$$

- The coefficients above differ from the corresponding non-relativistic coefficients.

Two-Body Problem

$$\langle m_{12}, j_{12}; \vec{p}_{12}, \mu_{12}; l_{12}, s_{12} | V | m'_{12}, j'_{12}; \vec{p}'_{12}, \mu'_{12}; l'_{12}, s'_{12} \rangle =$$

$$\delta(\vec{p}_{12} - \vec{p}'_{12}) \delta_{j_{12} j'_{12}} \delta_{\mu_{12} \mu'_{12}} \langle m_{12}, l_{12}, s_{12} \| V^j \| m'_{12}, l'_{12}, s'_{12} \rangle$$

$$\Downarrow$$

$$|M, j_{12}; \vec{p}_{12}, \mu_{12}\rangle$$

Two-Body Problem

$$|M, j_{12}; \vec{0}_{12}, j_{12}\rangle$$

$$\Downarrow \qquad (j_{12-})$$

$$|M, j_{12}; \vec{0}_{12}, \mu_{12}\rangle$$

$$\Downarrow \qquad (U(\Lambda(p))$$

$$|M, j_{12}; \vec{p}_{12}, \mu_{12}\rangle$$

The Two-Body Problem

$$\Downarrow$$

$$U_I(\Lambda, a)|M, j_{12}; \vec{p}_{12}, \mu_{12}\rangle$$

- This construction leads to a $U(\Lambda, a)$ for the interacting system on the two body Hilbert space.
- The dynamical problem is to diagonalize $M = m_{12} + V$ in the basis of free particle eigenstates.

Beyond-Two

- The construction has been generalized to treat any number of particle and with limited particle production.
- The requirement of cluster properties adds new dynamical non-linear constraints for more than two particles. The have been treated.
- $U(\Lambda, a)$ implies that spinor, tensor, and four vector operators are interaction dependent

Conclusions

- Relativistic quantum mechanics of particles is a useful and practical tool for modeling quantum systems where relativity is important.
- It has been successfully applies to model spectral, electromagnetic, and scaling properties of baryons, mesons, and the two and three nucleon systems.
- Open question on how to treat problems with particle production are being explored