

Poincaré invariant quantum mechanics: an alternative to integrating relativity with quantum mechanics

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Introduction to Quantum Theory

- Complex vector space
- Scalar product

$$(\vec{a})^* \cdot \vec{b} := \langle a|b \rangle$$

Example: electron at rest

- **Basis vectors**

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- **Scalar product**

$$\langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = 1 \quad \langle\downarrow|\uparrow\rangle = \langle\uparrow|\downarrow\rangle = 0$$

Measurements

The state of an electron is represented by a unit vector

$$|e\rangle = \cos(\theta) |\uparrow\rangle + \sin(\theta) |\downarrow\rangle$$

$$P_{\uparrow} := |\langle e | \uparrow \rangle|^2 = |\cos(\theta)|^2 = \cos^2(\theta)$$

$$P_{\downarrow} := |\langle e | \downarrow \rangle|^2 = |\sin(\theta)|^2 = \sin^2(\theta)$$

$$P_{\downarrow} + P_{\uparrow} = \sin^2(\theta) + \cos^2(\theta) = 1$$

$$P_{a,b} = |\langle a|b\rangle|^2$$

Key Observations

- **The Schrödinger equation is not needed to discuss quantum measurements.**
- **The result of any measurement is a probability.**

Symmetries

A vector correspondence $|a\rangle \rightarrow |a'\rangle$ is a **symmetry** of a quantum theory if

$$P_{ab} = |\langle a|b\rangle|^2 = |\langle a'|b'\rangle|^2 = P_{a'b'}$$

If $|a\rangle \rightarrow |a'\rangle$ is a Symmetry



Physics in the unprimed world is **indistinguishable** from physics in the primed world.

Relativity = existence of inertial reference frames

Physics in different inertial reference frames is indistinguishable

$$|a\rangle \rightarrow |a'\rangle$$

\Downarrow

$$P_{ab} = P_{a'b'}$$

Differs from the historical development

How are inertial reference frames related?

Use classical physics

- By coordinate transforms that preserve the form of Newton's second law ? (**Galilean relativity**)
- By coordinate transforms that preserve the form of Maxwell's equations of electricity and magnetism ? (**Special relativity**)

Galilean Relativity preserves

$$\frac{d^2\vec{x}}{dt'^2} = 0 \quad \Leftrightarrow \quad \frac{d^2\vec{x}'}{dt'^2} = 0$$

$$\vec{x}' = R\vec{x} + \vec{v}t + \vec{a} \quad t' = t + c$$

Special Relativity preserves

$$|\vec{x} - \vec{y}|^2 - c^2|t_x - t_y|^2 = |\vec{x}' - \vec{y}'|^2 - c^2|t'_x - t'_y|^2$$

$c =$ speed of light

Which is the correct symmetry of nature?

Michelson-Morley Experiment



Special relativity

Poincaré Transformations

$$|\vec{x} - \vec{y}|^2 - c^2|t_x - t_y|^2 = |\vec{x}' - \vec{y}'|^2 - c^2|t'_x - t'_y|^2$$

$$x^\mu := (ct, x^1, x^2, x^3)$$

Differentiate above $\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial y^\nu}$

$$\Lambda^\mu{}_\nu := \frac{\partial x'^\mu}{\partial x^\nu}(0) \quad a^\mu = x'^\mu(0)$$

\Downarrow

$$x'^\mu = \sum_{\nu=0}^3 \Lambda^\mu{}_\nu x^\nu + a^\mu \quad \Lambda_\mu{}^\alpha \eta^{\mu\nu} \Lambda_\nu{}^\beta = \eta^{\alpha\beta}$$

$$\eta^{\mu\nu} := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note the similarity to

$$\vec{x}' = O\vec{x} + \vec{a} \quad Q^t I O = I$$

which are the corresponding equations for rigid-body motion

Basic Poincaré Transformations

Rotations (3):

$$\vec{x}' = O\vec{x} \quad O^t O = I$$

Translations (4):

$$\vec{x}' = \vec{x} + \vec{a}, \quad ct' = ct + a^0$$

Rotationless Lorentz transformations (3):

$$x' = \gamma(x + vt) \quad t' = \gamma(t + vx/c^2) \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

All Poincaré transformations can be generated from these ten elementary transformations

All 10 elementary Poincaré transformations have inverses

$$O \rightarrow O^t$$

$$\vec{a} \rightarrow -\vec{a}; \quad a^0 \rightarrow -a^0$$

$$V \rightarrow -V$$

$$x' = \Lambda_1 x + a_1$$

$$x'' = \Lambda_2 x' + a_2$$

↓

$$x'' = (\Lambda_2 \Lambda_1) x + (\Lambda_2 a_1 + a_2)$$

$$\Lambda_3 := \Lambda_2 \Lambda_1 \quad a_3 = \Lambda_2 a_1 + a_2$$

↓

$$x'' = \Lambda_3 x + a_3$$

- $\Lambda^\mu{}_\nu$ and a^μ are labels for distinct inertial reference frames.
- All Poincaré transformations can be generated from 10 elementary transformations.
- Compositions of Poincaré transformations are Poincaré transforms.
- Poincaré transformations are elements of a group (closed under composition, inverse, identity, associative).

Special relativity and quantum mechanics

The group of Poincaré transformations is a symmetry of quantum mechanics

$$|v\rangle \rightarrow |v_{\Lambda,a}\rangle$$

\Downarrow

$$P_{uv} = P_{u_{\Lambda,a},v_{\Lambda,a}}$$

Eugene P. Wigner

$$P_{uv} = P_{u_{\Lambda,a} v_{\Lambda,a}}$$

↓

$$\langle u_{\Lambda,a} | v_{\Lambda,a} \rangle = \langle u | v \rangle$$

$$U(\Lambda, a) | v \rangle = | v_{\Lambda,a} \rangle$$

$$U := U(\Lambda, a) \quad U^\dagger U = (U^t)^* U = I$$

$$U_3 | v \rangle = U_2 | v_1 \rangle = U_2 U_1 | v \rangle$$

↓

$$U(\Lambda_2, a_2) U(\Lambda_1, a_1) = U(\Lambda_2 \Lambda_1, \Lambda_2 a_1 + a_2)$$

Note that $U(\Lambda, a)$ has the multiplication properties of the Poincaré group but it acts on a vector space with different dimension than four.

$U(\Lambda, a)$ is a unitary **representation** of the Poincaré group.

How do we construct $U(\Lambda, a)$?

Rotating an electron at rest

$$U(R)|\uparrow\rangle = |\uparrow\rangle u_{\uparrow\uparrow} + |\downarrow\rangle u_{\downarrow\uparrow}$$

$$U(R)|\downarrow\rangle = |\uparrow\rangle u_{\uparrow\downarrow} + |\downarrow\rangle u_{\downarrow\downarrow}$$

$$U(R)|\mu\rangle = \sum_{\nu=\uparrow,\downarrow} |\nu\rangle D_{\nu\mu}^2(R)$$

$D_{\nu\mu}^2(R)$ is a two dimensional representation of the rotation group.

Translating and electron at rest

$$U(a^0, \vec{a})|\mu\rangle = |\mu\rangle e^{-imca^0}$$

Electron moving with velocity \vec{v} , ($\vec{p} = m_e \gamma \vec{v}$)

$$|\vec{p}, \mu\rangle := U(\vec{v})|\mu\rangle$$

Every Poincaré transformation can be expressed as

- Rotationless Lorentz transform to rest frame
- Rotation
- Translation of a rest state
- Rotationless Lorentz transform to final frame

$$U(\Lambda, a)|\vec{p}, \mu\rangle = \sum_{\mu'} |\vec{p}', \mu'\rangle D_{\mu'\mu}^2(R(\Lambda, p)) e^{-ip' \cdot a}$$

This transformation law is a mass m_e spin $1/2$ irreducible representation for the Poincaré group.

This can be done for particles with any mass and spin.

Two free electrons in zero total momentum frame

$$|\vec{k}, \mu_1; -\vec{k}, \mu_2\rangle := |\vec{0}, \vec{k}, \mu_1, \mu_2\rangle =$$

- Decompose into linear combinations that have definite angular momenta, J
- Resulting states look just like free particle states with mass and spin

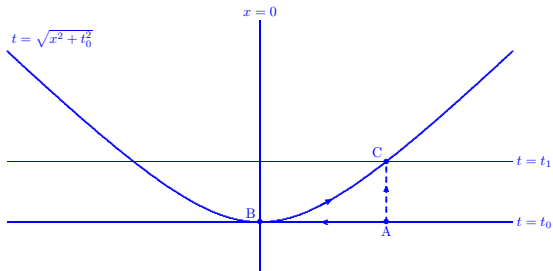
$$M = 2\sqrt{k^2 + m_e^2} \quad J$$

- Repeat construction to make irreducible two particle states

$$|\vec{p}, \mu(M(k), J, d)\rangle$$
$$U(\Lambda, a)|\vec{p}, \mu(M(k), J, d)\rangle =$$
$$\sum_{\mu'} |\vec{p}', \mu'(M(k), J, d)\rangle D_{\mu'\mu}^{2J+1}(R(\Lambda, p)) e^{-ip'\cdot a}$$

Two Interacting Electrons

Consistent initial value problem?



Solving the non-linear problem?

- Add interactions to mass

$$M = 2\sqrt{k^2 + m_1^2} + V$$

- Make sure that V it does not mix different angular momenta
- Make sure that V is independent of the quantities that change under change of inertial coordinate system.
- Diagonalize the matrix M in the free two electron states.

$$(2\sqrt{k^2 + m_e^2} + V)|\vec{p}, \mu, (M', J)\rangle = M'|\vec{p}, \mu, (M', J)\rangle$$

$$U_I(\Lambda, a)|\vec{p}, \mu(M', J)\rangle =$$

$$\sum_{\mu'} |\vec{p}', \mu'(M', J)\rangle D_{\mu'\mu}^{2J+1}(R(\Lambda, p)) e^{-ip' \cdot a}$$

This defines a relativistic interacting two-electron quantum theory

Conclusion

- The realization of special relativity in quantum mechanics is very different than it is in classical physics.
- The Poincaré group provides alternate paths to the future.
- The Poincaré symmetry can be preserved with interactions if they are added to the two-body invariant mass.
- Framework is often used in nuclear physics.