Hamiltonian formulation of quantum field theory using wavelets

W. N. Polyzou

Other contributors: Gerald Payne, Brian Kessler, Fatih Bulut, Tracie Michlin

The University of Iowa

Hamiltonian Field Theory for QCD and Hadron Physics
Granada 2023

U.S. DEPARTMENT OF ENERGY | Office of Science

Hamiltonian methods (advantages)

- Non-perturbative formulations possible.

- Non-perturbative dynamics reduces to linear algebra - many available computational methods - variational, Krylov, $\cdots$.

- Natural for quantum computing - real time path integrals.

Field theory challenges

- Hamiltonians involve ill-defined products of operator-valued distributions.
- The Stone-Von Neumann theorem does not hold for theories with an infinite number of degrees of freedom (inequivalent Hilbert space representations).
- Non-perturbative renormalization required for non-perturbative calculations.

Relevant Observations

- **Reactions take place in a finite volume.**
- **The available energy in a reaction is finite; this limits the accessible resolution.**
- **Locally compact Hamiltonians:** $\Pi(|\mathbf{r}| < R)\Pi(H < E)$ **compact** $\to$ **a finite number of degrees of freedom are accessible (**$\approx$ **accessible volume in phase space divided by powers of h** $< \infty$**).**
- **In principle any reaction can be expressed in terms of a Hamiltonian involving these accessible degrees of freedom (**$\{\pi_n, \phi_n\}$**).**

Wavelet representation of local quantum field theory - takes advantage of these observations

- Formally exact representation of the field theory.
- Operator valued distributions are replaced by infinite expansions in well-defined almost local operators .
- The representation has natural volume and resolution truncations. The Stone Von-Neumann theorem applies to the truncations.
- Hamiltonians truncated to different resolutions are self similar.
- The interactions are almost local.

Daubechies wavelets and scaling functions

- **Orthonormal basis of functions on the real line satisfying:**
  - **Each basis function has compact support.**
  - **There are an infinite number of basis functions with support in any open set.**
  - **The basis functions have some smoothness.**
  - **The basis functions are generated from the solution of a linear renormalization group equation by translations and scale transformations (they are fractal valued).**
  - **The basis functions can be used to construct exact, locally finite representations of low-degree polynomials.**
  - **The basis provides an efficient representation of operators (used for data compression in jpeg and FBI fingerprint database)**
  - **Subsets of basis functions define locally finite partitions of unity.**

Ingrid Daubechies

Basis construction

s(x) fixed point of the renormalization group equation.

$$s(x) = \underbrace{D\ (\underbrace{\sum_{l=0}^{2K-1} h_l T^l s(x)}_{\text{weighted average}})}_{\text{rescale}}\ .$$

The solutions $s(x)$ are fractal!

T**: unit translation** D**: dyadic scale transformation**

$$Ts(x) = s(x-1) \qquad Ds(x) = \sqrt{2}s(2x).$$

The renormalization group equation is homogeneous.

scale of $s(x)$ is fixed by $\int dx s(x) = 1$

The weights, h_l, are determined by the following three the properties of $s(x)$

$$\sum_{l=1}^{2K-1} h_l = \sqrt{2}$$ **needed for solution of the RG equation**

Orthonormality of integer translates

$$\int s(x-n)s(x-m)dx = \delta_{mn}$$

Local pointwise representation of low-degree polynomials (depends on K)

$$x^m = \sum c_n s(x-n) \qquad m < K$$

These conditions define Daubechies K scaling function $s(x)$. They can be used to solve for the h_l.

Weight coefficients h_l for different K values

h_l	K=1	K=2	K=3
h_0	$1/\sqrt{2}$	$(1+\sqrt{3})/4\sqrt{2}$	$(1+\sqrt{10}+\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_1	$1/\sqrt{2}$	$(3+\sqrt{3})/4\sqrt{2}$	$(5+\sqrt{10}+3\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_2	0	$(3-\sqrt{3})/4\sqrt{2}$	$(10-2\sqrt{10}+2\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_3	0	$(1-\sqrt{3})/4\sqrt{2}$	$(10-2\sqrt{10}-2\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_4	0	0	$(5+\sqrt{10}-3\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_5	0	0	$(1+\sqrt{10}-\sqrt{5+2\sqrt{10}})/16\sqrt{2}$

Properties of $s(x)$ for K=3

Support $(s(x)) = [0,5]$ - **compact.**

$s(x)$ **has one continuous derivative.**

Fourier transform of $(s(x))$ **is entire.**

$1 = \sum_n s(x-n)$ **(locally finite partition of unity).**

$$\int s(x)dx = \int s(x)^2 dx = 1 \qquad \int s(x)s(x-n)dx = \delta_{0n}$$

$$x = \sum_n a_n s(x-n) \qquad x^2 = \sum_n b_n s(x-n)$$

$\langle x^m \rangle := \int x^m s(x)dx$ **exact expressions in terms of the** h_l**.**

Multi-resolution decomposition of $L^2(\mathbb{R})$

Re-scale and translate the fixed point, $s(x)$

$$s_n^k(x) := D^k T^n s(x) = 2^{k/2} s\left(2^k(x - 2^{-k}n)\right).$$

$$\textbf{support}(s_n^k(x)) = [2^{-k}n, 2^{-k}(n+5)]$$

$$\mathcal{S}_k := \{f(x) | f(x) = \sum_{n=-\infty}^{\infty} f_n s_n^k(x), \quad \sum_{n=-\infty}^{\infty} |f_n|^2 < \infty\}.$$

$\mathcal{S}_k :=$ **resolution 2^{-k} subspace of $L^2(\mathbb{R})$,** $\{s_n^k(x)\}$ **basis**

RG equation: $$s_n^k(x) = \sum_l h_l s_{2n+l}^{k+1}(x)$$

$$\mathcal{S}_k \subset \mathcal{S}_{k+1}$$

$$\mathcal{S}_{k+1} = \mathcal{S}_k \oplus \mathcal{W}_k \qquad \mathcal{W}_k \neq \{\emptyset\}.$$

Multi-resolution decomposition of $L^2(\mathbb{R})$

$$\mathcal{S}_{k+1} = \mathcal{S}_k \oplus \mathcal{W}_k$$

$$L^2(\mathbb{R}) = \mathcal{S}_k \oplus \mathcal{W}_k \oplus \mathcal{W}_{k+1} \oplus \mathcal{W}_{k+2} \oplus \mathcal{W}_{k+3} \oplus \cdots =$$

$$\cdots \oplus \mathcal{W}_{k-2} \oplus \mathcal{W}_{k-1} \oplus \mathcal{W}_k \oplus \mathcal{W}_{k+1} \oplus \mathcal{W}_{k+2} \oplus \cdots$$

Wavelets **(**$\{w_n^k(x)\}$ **orthonormal basis for** $\mathcal{W}_k$**)**

$$w(x) := D(\sum_{l=0}^{2K-1} g_l T^l s(x)) \qquad g_l = (-)^l h_{2K-1-l}$$

$$w_n^k(x) := D^k T^n w(x) = 2^{k/2} w\left(2^k(x - 2^{-k}n)\right).$$

Multi-resolution orthonormal bases

Orthonormal bases for $L^2(\mathbb{R})$:

$$\{s_n^k(x)\}_{n=-\infty}^{\infty}\} \cup \{w_n^{k+l}(x)\}_{n=-\infty,l=0}^{\infty}$$

or

$$\{w_n^l(x)\}_{n,l=-\infty}^{\infty}$$

Orthonormal basis for $L^2(\mathbb{R}^3)$:

$$\xi_n(x) \in \{s_n^k(x), w_n^l(x)\} \qquad \boldsymbol{\xi_{\mathbf{n}}(\mathbf{x})} := \xi_{n_1}(x)\xi_{n_2}(y)\xi_{n_3}(z)$$

Interpretation

$\mathcal{S}_k$ **subspace of square integrable functions with resolution** $\frac{5}{2^k}$

$\mathcal{W}_k$ **subspace adds details between resolution** $\frac{5}{2^k}$ **and resolution** $\frac{5}{2^{k+1}}$

Remark

$$\int x^m w_n^k(x)dx = 0 \qquad m < K$$

Wavelet Transform

$$\mathcal{S}_{k+n} = \mathcal{S}_k \otimes_{m=k}^{n+k-1} \mathcal{W}_m$$

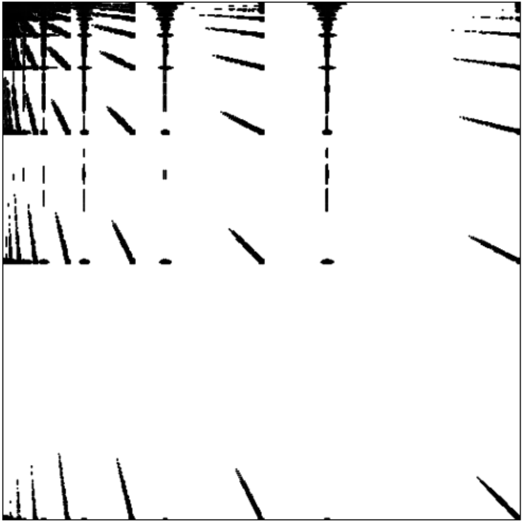
Orthogonal transformation - relates fine scale basis to multiple scale basis.

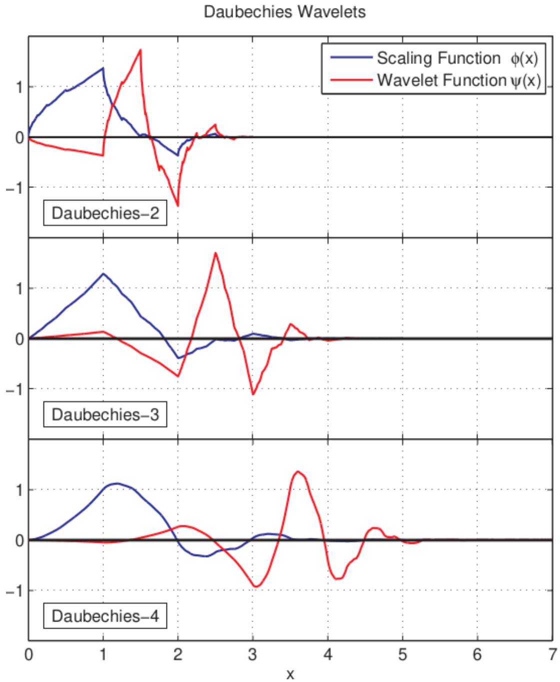
Orthogonal transformation: $O(N)$, 4K$\times$N additions

Multi-scale basis gives sparse representation of operators

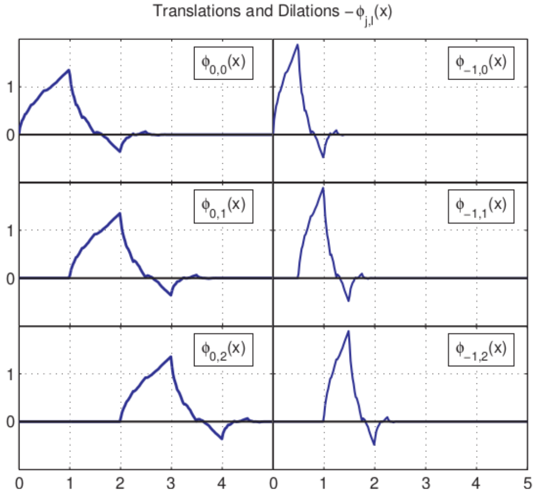
Kernel, $K(x, y)$ of a scattering integral equation after wavelet transform

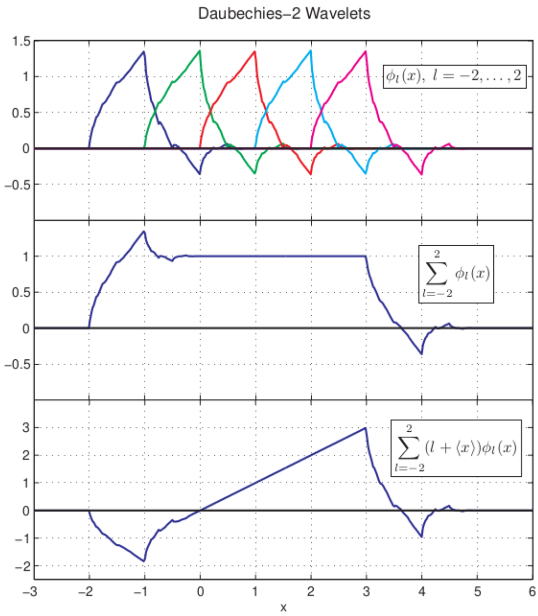
$$F(x) = D(x) + \int K(x, y)F(y)dy$$





Daubechies Wavelets
Scaling Function φ(x)
Wavelet Function ψ(x)
Daubechies−2
Daubechies−3
Daubechies−4
1
0
−1
0
1
2
3
4
5
6
7
x


Translations and Dilations – $\phi_{j,l}(x)$
$\phi_{0,0}(x)$
$\phi_{-1,0}(x)$
$\phi_{0,1}(x)$
$\phi_{-1,1}(x)$
$\phi_{0,2}(x)$
$\phi_{-1,2}(x)$
1
0
0 1 2 3 4
0 1 2 3 4 5



Daubechies−2 Wavelets
$\phi_l(x),\ l=-2,\ldots,2$
$\sum_{l=-2}^{2}\phi_l(x)$
$\sum_{l=-2}^{2}(l+\langle x\rangle)\phi_l(x)$
x

Quantum fields

Exact multi-resolution decomposition of field operators

$$\Phi(\mathbf{x},t) = \sum_{\mathbf{n}} \Phi^k_{\mathbf{n}}(t)\xi_{\mathbf{n}}(\mathbf{x}) \qquad \Phi^k_{\mathbf{n}}(t) = \int d\mathbf{x}\xi_{\mathbf{n}}(\mathbf{x})\Phi(\mathbf{x},t)$$

$$\Pi(\mathbf{x},t) = \sum_{\mathbf{n}} \Pi^k_{n}(t)\xi_{\mathbf{n}}(\mathbf{x}) \qquad \Pi^k_{\mathbf{n}}(t) = \int d\mathbf{x}\xi_{\mathbf{n}}(\mathbf{x})\Pi(\mathbf{x},t)$$

Canonical commutation relations

$$[\Phi_{\mathbf{n}}(t), \Pi_{\mathbf{m}}(t)] = i\delta_{\mathbf{n},\mathbf{m}}$$

$$[\Phi_{\mathbf{n}}(t), \Phi_{\mathbf{m}}(t)] = [\Pi_{\mathbf{n}}(t), \Pi_{\mathbf{m}}(t)] = 0,$$

Interpretation: $\Phi_{\mathbf{n}}(t)$ **average of** $\Phi(\mathbf{x},t)$ **over support of** $\xi_{\mathbf{n}}$ **.**

- **Expansion is exact.**
- **Operator valued distributions are replaced by infinite sums of well-defined discrete field operators.**
- **Products of discrete fields are well defined.**
- **The discrete field operators are labeled by position and resolution.**
- **Discrete fields satisfy canonical commutation relations.**
- **Exact decomposition of the field into localized observables by resolution.**
- **Natural resolution (limit l) and volume (limit n) truncations ($\{s_n^k(x), w_n^{k+l}(x)\}$).**

Example

$\phi^4(x)$ Hamiltonian - discrete representation formally exact

$$H = \frac{1}{2}\sum_{\mathbf{m}} \hat{\Pi}^2_{\mathbf{m}} + \frac{1}{2}\sum_{\mathbf{mn}} D_{\mathbf{m},\mathbf{n}} \hat{\Phi}_{\mathbf{m}} \hat{\Phi}_{\mathbf{n}} + \frac{\mu^2}{2}\sum_{\mathbf{m}} \hat{\Phi}^2_{\mathbf{m}}$$

$$+\frac{\lambda}{2}\sum_{\mathbf{m}_1,\mathbf{m}_2\mathbf{m}_3,\mathbf{m}_4} \Gamma_{\mathbf{m}_1,\mathbf{m}_2\mathbf{m}_3,\mathbf{m}_4} \hat{\Phi}_{\mathbf{m}_1} \hat{\Phi}_{\mathbf{m}_2} \hat{\Phi}_{\mathbf{m}_3} \hat{\Phi}_{\mathbf{m}_4}$$

where

$$\xi_n(x) \in \{s^k_n(x), w^l_n(x)\}$$

$$D_{\mathbf{m},\mathbf{n}} := \int \nabla \xi_{\mathbf{m}}(\mathbf{x}) \cdot \nabla \xi_{\mathbf{n}}(\mathbf{x}) d\mathbf{x}$$

$$\Gamma_{\mathbf{m}_1,\cdots,\mathbf{m}_n} := \int \xi_{\mathbf{m}_1}(\mathbf{x}) \cdots \xi_{\mathbf{m}_n}(\mathbf{x}) d\mathbf{x}$$

Infinities due to non-convergence of infinite sums.

The numerical coefficients can be computed exactly.

The resolution k coefficients are related to resolution zero coefficients by scale transformations and translations:

$$D^k_{mn} = 2^{2k} D^0_{0,n-m} \qquad \Gamma^k_{m_1,\cdots,m_n} = 2^{k(n/2-1)} \Gamma^0_{0,m_2-m_1,\cdots m_n-m_1}$$

these vanish unless $|m|, |m_i - m_j| \leq 4$ for $K = 3$.

The non-zero $D^0_{0,n}$ and $\Gamma^0_{0,m_2,\cdots,m_n}$ are solutions of a finite linear systems derived using the renormalization group equation and the scale fixing condition.

Example: $D_{\mathbf{mn}}$

zero unless the support of s_m and s_n overlap

$$D_{mn} = D_{m-n,0} = \int \frac{ds_m(x)}{dx}\frac{ds_n(x)}{dx}dx$$

Non-zero solutions have exact rational values

$$D_{40} = D_{-40} = -3/560$$

$$D_{30} = D_{-30} = -4/35$$

$$D_{20} = D_{-20} = 92/105$$

$$D_{10} = D_{-10} = -356/105$$

$$D_{00} = 295/56.$$

The $\Gamma^0_{0,m_2-m_1,\cdots m_n-m_1}$ can be solved by the same methods, but there are more of them.

Heisenberg picture

$$\dot{\Phi}_{\mathbf{n}}(t) = i[H, \Phi_{\mathbf{n}}(t)] = \Pi_{\mathbf{n}}(t)$$

$$\dot{\Pi}_{\mathbf{n}}(t) = i[H, \Pi_{\mathbf{n}}(t)] = \mu^2 \Phi_{\mathbf{n}}(t) + \sum_{\mathbf{nm}} D_{\mathbf{0m-n}} \Phi_{\mathbf{m}}(t)$$

$$+4\lambda \sum_{\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3} \Gamma_{\mathbf{0},\mathbf{m}_1-\mathbf{n},\mathbf{m}_2-\mathbf{n},\mathbf{m}-\mathbf{n}} \Phi_{\mathbf{m}_1}(t) \Phi_{\mathbf{m}_2}(t) \Phi_{\mathbf{m}_3}(t)$$

This is an infinite number of non-linear coupled equations for

$\Phi_{\mathbf{m}}(t)$ and $\Pi_{\mathbf{m}}(t)$

Discrete many-body quantum theory

Schrodinger picture - linear

$$\Psi[\boldsymbol{\phi}] := \Psi[\phi_{\mathbf{n}_1}, \cdots, \phi_{\mathbf{n}_N}]$$

$$H = (-\sum_n \frac{\partial^2}{\partial \phi_n^2} + V[\boldsymbol{\phi}])\Psi[\boldsymbol{\phi}] = E\Psi[\boldsymbol{\phi}]$$

$$i\frac{\partial}{\partial t}\Psi(\boldsymbol{\phi}, t) = (-\sum_n \frac{\partial^2}{\partial \phi_n^2} + V[\boldsymbol{\phi}])\Psi[\boldsymbol{\phi}, t]$$

$$V[\boldsymbol{\phi}] = \sum_n \mu^2 {\phi_n}_{\mathbf{n}}^2 + \sum_{\mathbf{nm}} D_{\mathbf{nm}}\phi_{\mathbf{n}}\phi_{\mathbf{m}}$$

$$+\lambda \sum_{\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3,\mathbf{m}_4} \Gamma_{\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3,\mathbf{m}_4}\phi_{\mathbf{m}_1}\phi_{\mathbf{m}_2}\phi_{\mathbf{m}_3}\phi_{\mathbf{m}_4}$$

Non-perturbative - use Trotter product formula (path integral)

$$e^{-iHt} = \lim_{N\to\infty} [e^{iH_0 t/N} e^{iVt/N}]^N$$

$$H_0 = \frac{1}{2}\sum_{\mathbf{m}} \Pi^2_{\mathbf{m}}$$

$$V = \frac{1}{2}\sum_{\mathbf{mn}} D_{\mathbf{m},\mathbf{n}}\Phi_{\mathbf{m}}\Phi_{\mathbf{n}} + \frac{\mu^2}{2}\sum_{\mathbf{m}} \Phi^2_{\mathbf{m}}$$

$$+\frac{\lambda}{2}\sum_{\mathbf{m}_1,\mathbf{m}_2\mathbf{m}_3,\mathbf{m}_4} \Gamma_{\mathbf{m}_1,\mathbf{m}_2\mathbf{m}_3,\mathbf{m}_4}\Phi_{\mathbf{m}_1}\Phi_{\mathbf{m}_2}\Phi_{\mathbf{m}_3}\Phi_{\mathbf{m}_4}$$

Example (Nathanson-Jørgensen)

$$\langle \phi_{1f}, \phi_{2f} | e^{-iHt} | \phi_{1i}, \phi_{2i} \rangle = \sum_{\text{paths}} P[path] e^{iV[path,t]}$$

$P =$ **complex probability on cylinder sets of paths;** $V[path, t]$ **potential functional of paths (J. Math. Phys. 56,092102(2015)).**

$\phi^4(x)$**: 2 degrees of freedom** ϕ_0, ϕ_1

41 field amplitudes each ϕ_0, ϕ_1

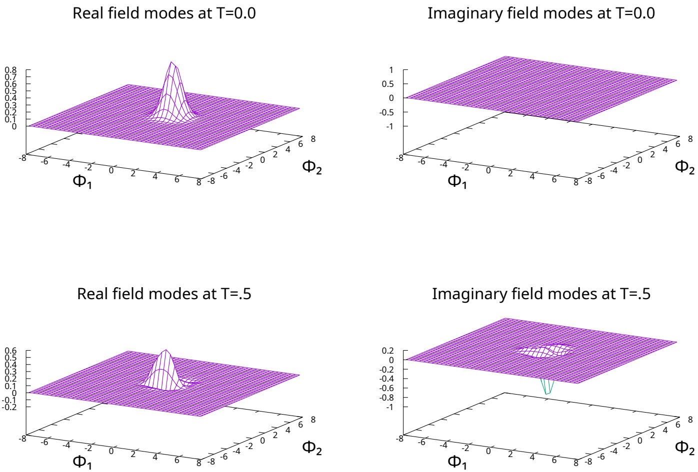
20 time steps

t=.5

$(41 \times 41)^{20}$ **cylinder sets of paths**

$P[path]e^{iV[path,t]}$ **factors into product of unitary operators**

$$\langle \phi_{1f}, \phi_{2f} | e^{-iHt} | \phi_{1i}, \phi_{2i} \rangle = \sum_{\text{paths}} P[path] e^{iV[path,t]}$$



Real field modes at T=0.0
Imaginary field modes at T=0.0
Real field modes at T=.5
Imaginary field modes at T=.5
Φ_1
Φ_2

Non-perturbative renormalization

Truncated Hamiltonians with different resolutions are self-similar

$$H = \frac{1}{2}\sum_{\mathbf{m}} \hat{\Pi}^2_{\mathbf{m}} + \frac{1}{2}\sum_{\mathbf{mn}} 2^{2k} D_{\mathbf{m},\mathbf{n}} \hat{\Phi}_{\mathbf{m}}\hat{\Phi}_{\mathbf{n}} + \frac{\mu^2}{2}\sum_{\mathbf{m}} \hat{\Phi}^2_{\mathbf{m}}$$

$$+\frac{\lambda}{2} 2^{k(n/2)-1} \sum_{\mathbf{m}_1,\mathbf{m}_2\mathbf{m}_3,\mathbf{m}_4} \Gamma_{\mathbf{m}_1,\mathbf{m}_2\mathbf{m}_3,\mathbf{m}_4} \hat{\Phi}_{\mathbf{m}_1}\hat{\Phi}_{\mathbf{m}_2}\hat{\Phi}_{\mathbf{m}_3}\hat{\Phi}_{\mathbf{m}_4}$$

Functional renormalization group equation relates Hamiltonians with different resolutions:

$$H_k(\Pi, \Phi, \mu, \lambda) = 2^k H_0(2^{-k}\Pi, 2^k\Phi, 2^{-2k}\mu, 2^{-2k}\lambda]$$

Wave function, mass and coupling constant renormalization; canonical transformation.

Application

Pick a starting (physical) scale - adjust parameters in H to fix observables at that scale.

Increase resolution - readjust parameters H to fix observables at the physical scale.

Block diagonalize Hamiltonian by scale to decouple fine scale degrees of freedom from physical degrees of freedom.

Gives effective Hamiltonian in terms of physical scale degrees of freedom.

Example - free field - derivatives couple scales

Use flow equation method (Glazek-Wilson) to block diagonalize Hamiltonian by scale (Michlin).

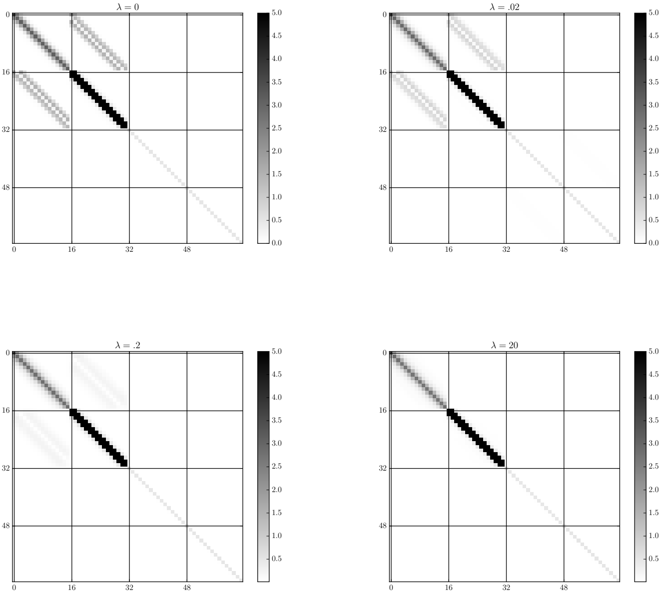
$$U(\lambda) = e^{iF(\lambda)}, \qquad F(\lambda) = -i[G(\lambda), H(\lambda)]$$

where $G(\lambda)$ is the part of $H(\lambda)$ with the operators that couple different scales turned off. With this choice $G(\lambda) = G^\dagger(\lambda)$ so $F(\lambda)$ is Hermitian.

It follows that

$$\frac{dH(\lambda)}{d\lambda} = i[F(\lambda), H(\lambda)] = -i[H(\lambda), [H(\lambda), G(\lambda)]].$$

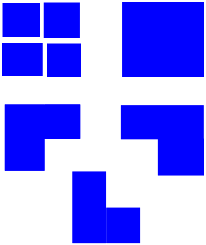
$H(\lambda)$ is block diagonal when $[H(\lambda), G(\lambda)] = 0$.



$\lambda = 0$
$\lambda = .02$
$\lambda = .2$
$\lambda = 20$
0
16
32
48
5.0
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0

Speculations?

Formally exact locally gauge invariant formulation of QCD?
Irreducible multi-resolution set of gauge invariant operators?



For non-Abeilan case we need to treat traces (Mandelstam identities - Phys. Rev. D19(1979)2391 $SU(2)$ case)

$$\delta_{ij}\delta_{kl} = \delta_{il}\delta_{jk} + \sum_m \epsilon_{ikm}\epsilon_{jlm}$$

Summary - Wavelet representation

- Formally exact representation of field theories.
- Hamiltonian form non-perturbative.
- Natural volume and resolution truncations.
- Truncated fields have continuous x dependence.
- Product of distrubutions replaced by products of operators.
- Poincaré generators truncated to resolution (violations at higher resolution).
- Light-front formulation for applications to proton tomography.
- Thanks: Organizers and Audience!

References

- W. N. Polyzou, Path integrals and the discrete Weyl representation, arxiv:2108.12494.
- W. N. Polyzou, Wavelet representation of light-front quantum field theory, Phys. Rev. D, 101,064001(2020), arxiv:abs/2002.02311.
- Tracie Michlin, W. N. Polyzou and Fatih Bulut, Multiresolution decomposition of quantum field theories using wavelet bases, Phys. Rev. D95 (2017) no.9, 094501, arxiv:1601.08612.
- Fatih Bulut and W. N. Polyzou, Wavelets in Field Theory, Phys. Rev. D87, 116011 (2013), arXiv:1301.6570.
- Fatih Bulut and W. N. Polyzou, Wavelet Methods in the Relativistic Three-Body Problem, Phys. Rev. C73,e024003(2006), nucl-th/0507051. pdf, ps
- B. Kessler, G. L. Payne, W. N. Polyzou Application of wavelets to singular integral scattering equations, nucl-th/0406079. Phys. Rev. C70,034003(2004)
- B. M. Kessler, G. L. Payne, W. N. Polyzou Scattering Calculations with Wavelets, Few-Body Systems, 33,1(2003), nucl-th/0211016.