Wavelets in Physics

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THE UNIVERSITY OF IOWA

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$$P(x,y) = \sum_{mn} c_{mn}\phi_m(x)\phi_n(y)$$

$c_{mn} =$
/211/212/222/223/224/225/226/227/230/231/232/242/234/244
/245/246/247/250/251/252/262/263/264/265/266/267/270/271
/272/302/303/304/305/306/307/310/311/312/322/323/324/325
/326/327/330/331/332/341/342/343/344/345/346/347/350/363
. . .

Outline

- Background
- Scaling functions
- Multiscale analysis and wavelets
- Numerical analysis with wavelets
- Applications to scattering

What are wavelets?

- **They are orthonormal basis functions,**

$$(\phi_m, \phi_n) = \int dx\, \phi_m(x)\phi_n(x) = \delta_{mn}.$$

- **They are used in signal processing and data compression algorithms.**
- **JPEG digital photo files are tables of expansion coefficients in a wavelet basis.**
- **FBI fingerprint files are stored as expansion coefficients in a wavelet basis.**

Interesting properties of wavelets

- The basis functions are related to fixed points of a renormalization group equation.
- The basis functions have a fractal structure, and are not amenable to standard numerical methods.
- Exact expansions of low-degree polynomials at any point only require a finite number of fractal basis functions.
- Wavelet bases are natural for problems where many scales are strongly coupled (quantum field theory, phase transitions).

My interest in wavelets?

- **The Schrödinger equation for few-body scattering problems can written in the form**

$$X(x) = B(x) + \int K(x,y)\,dy\,X(y) \qquad X(x) \textbf{ unknown}$$

where the operator K can be uniformly approximated by finite-dimensional matrices

$$K(x,y) \approx \sum_{mn}^{N} \psi_m(x) k_{mn} \psi_n(y) \qquad B(x) \approx \sum_{m=1}^{N} \psi_m(x) b_m$$

$$\Downarrow$$

$$x_m = b_m + \sum_{n=1}^{N} k_{mn} x_n \qquad X(x) \approx \sum_{m=1}^{N} \psi_m(x) x_m$$

For realistic problems the matrices k_{mn} are large (eg. $10^7 \times 10^7$ for three particle systems).

My interest in wavelets?

- k_{mn} is analogous to a digital photograph of $K(x, y)$.
- By using wavelets for $\psi_m(x)$ can we replace

$$x_m = b_m + \sum_{n=1}^{N} k_{mn} x_n$$

by an equation with smaller N with minimal loss of accuracy?

Elementary questions

How do we compute

$$\psi_m(x)$$

$$b_m = \int B(x)\, dx\, \psi_m(x)$$

$$k_{mn} = \int \psi_m(x)\, dx\, K(x,y)\, dy\, \psi_n(y)$$

when $\psi_m(x)$ is a fractal function?

Scaling functions
("Father wavelet")

Operators (unitary)

$$\underbrace{(Df)(x) = \sqrt{2}f(2x)}_{\textbf{scale change}\,(\frac{1}{2})} \qquad \underbrace{(Tf)(x) = f(x-1)}_{\textbf{translation}}.$$

Scaling equation

$$\phi(x) = D\left(\sum_{l=0}^{2K-1} h_l T^l \phi(x)\right) \qquad \int \phi(x)dx = 1$$

$\phi(x)$:= **Scaling function**

Renormalization group transformation

$$f'(x) = \underbrace{D\ \underbrace{\left(\sum_{l=0}^{2K-1} h_l T^l f(x)\right)}_{\text{weighted block average}}}_{\text{rescaling}\,(\frac{1}{2})}$$

The scaling function, $\phi(x)$, is a fixed point of the renormalization group transformation!

- h_l **are constant coefficients (called scaling coefficients) that determine the type of wavelet (we use Daubechies' wavelets).**

- K **is a finite integer.**

- **A necessary condition for the existence of a fixed point of the scaling equation is:**

$$\sum_{l=1}^{2K-1} h_l = \sqrt{2}.$$

Daubechies scaling coefficients?

h_n **constant coefficients satisfying**

$$\sum_{n=0}^{2K-1} h_n = \sqrt{2}$$

$$\sum_{n=0}^{2K-1} h_n h_{n-2m} = \delta_{m0}$$

$$g_n := (-l)^n h_{2K-1-n} \qquad \sum_{n=0}^{2K-1} n^m g_n = 0 \qquad m < K$$

Equations fix h_n **up to reflection,** $h_n \to h'_n = h_{2K-1-n}$

Daubechies' conditions on h_n

- **The first condition is necessary for a solution of the scaling equation.**

- **The second condition ensures that $\phi(x-m)$ and $\phi(x-n)$ are orthonormal.**

- **The third condition ensures that finite linear combinations of $T^n\phi = \phi(x-n)$ can exactly represent polynomials of degree $< K$.**

Daubechies' scaling coefficients, $K = 1, 2, 3$

h_l	**K=1**	**K=2**	**K=3**
h_0	$1/\sqrt{2}$	$(1+\sqrt{3})/4\sqrt{2}$	$(1+\sqrt{10}+\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_1	$1/\sqrt{2}$	$(3+\sqrt{3})/4\sqrt{2}$	$(5+\sqrt{10}+3\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_2	0	$(3-\sqrt{3})/4\sqrt{2}$	$(10-2\sqrt{10}+2\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_3	0	$(1-\sqrt{3})/4\sqrt{2}$	$(10-2\sqrt{10}-2\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_4	0	0	$(5+\sqrt{10}-3\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_5	0	0	$(1+\sqrt{10}-\sqrt{5+2\sqrt{10}})/16\sqrt{2}$

Computing $\phi(x)$

- **Theorem:** $[\phi(x)] = 0$ **unless** $0 < x < 2K - 1$
- **Theorem:** $K > 1 \quad \Rightarrow \quad \phi(x)$ **continuous.**

$$\phi(n) = \sqrt{2} \sum_{l=0}^{2K-1} \sqrt{2} h_l \phi(2n - l).$$

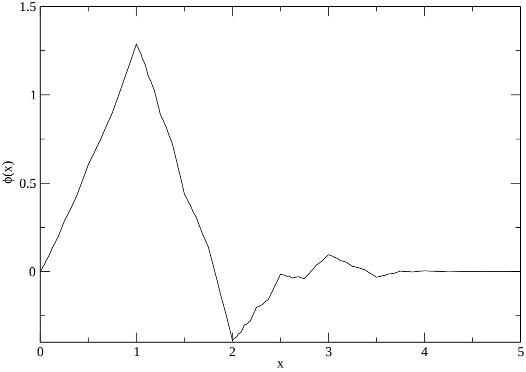
$$\sum_n \phi(n) = 1.$$

- **Solving above equations gives** $\phi(n)$, $n = 0, 1, 2, \cdots, 2K - 1$.

$$\phi(\frac{n}{2^{m+1}}) = \sum_{l=0}^{2K-1} \sqrt{2} h_l \phi(\frac{n}{2^m} - l).$$

- **Induction gives exact values of** $\phi(x)$ **at all dyadic rationals,** $n/2^m$**.**

Daubechies' K=3 scaling function



Properties of scaling function $\phi(x)$

- **Reality**

$$\phi(x) = \phi^*(x)$$

- **Partition of unity**

$$1 = \sum_{n=-\infty}^{\infty} \phi(x-n) = \sum_{n=-\infty}^{\infty} (T^n \phi)(x)$$

- **Compact support**

$$\phi(x) = 0 \qquad \textbf{unless} \qquad 0 < x < 2K - 1$$

- **Differentiability (K>2)**

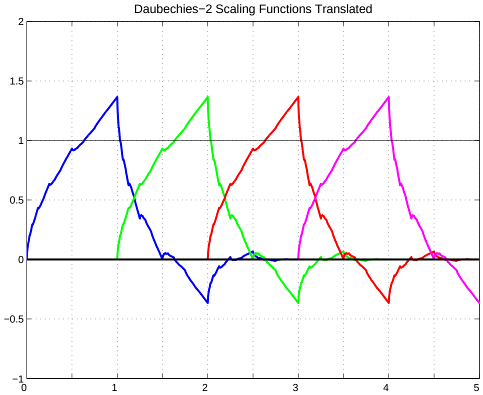
$$\frac{d\phi(x)}{dx} \qquad \textbf{exists} \qquad \textbf{for} \qquad K \geq 3$$

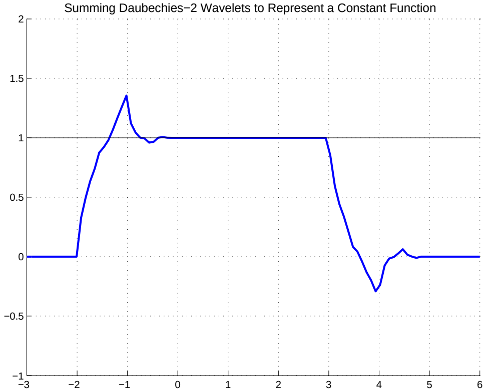
- **Orthonormality - scale fixing**

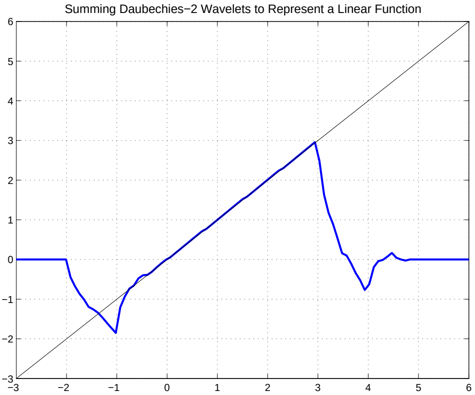
$$(T^m \phi, T^n \phi) = \delta_{mn} \qquad \int \phi(x) dx = 1$$

Approximation spaces

- $\phi_{mn}(x) := D^m T^n \phi(x)$.
- $(\phi_{mn}, \phi_{mk}) = \delta_{nk}$ **(fixed** m**)**.
- $\int \phi_{mn}(x)\,dx = \frac{1}{\sqrt{2^m}}$ $\phi_{mn}(x) = 0$ unless $2^{-m} l < x < 2^{-m}(l + 2K - 1)$
- $\{\phi_{mn}\}_n$ **can locally pointwise represent polynomials of degree** $< K$ **for each** m.
- $\mathcal{V}_m := \text{span}\{\phi_{mn}\}_n \cap L^2(\mathbb{R}) =$ "**scale** m **subspace**" **of** $L^2(\mathbb{R})$.


Daubechies−2 Scaling Functions Translated
2
1.5
1
0.5
0
−0.5
−1
0
1
2
3
4
5


Summing Daubechies−2 Wavelets to Represent a Constant Function
2
1.5
1
0.5
0
−0.5
−1
−3
−2
−1
0
1
2
3
4
5
6



Summing Daubechies−2 Wavelets to Represent a Linear Function
6
5
4
3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3
4
5
6

Wavelets

Multi-scale decomposition of $L^2(\mathbb{R})$

y

$$m > n \quad \Rightarrow \quad \mathcal{V}_m \supset \mathcal{V}_n$$

$$L^2(\mathbb{R}) \supset \cdots \supset \mathcal{V}_{n+1} \supset \mathcal{V}_n \supset \mathcal{V}_{n-1} \supset \cdots \supset \emptyset$$

$$\mathcal{V}_{n+1} = \mathcal{V}_n \oplus \mathcal{W}_n$$

$$\Downarrow$$

$$\mathcal{V}_n = \mathcal{W}_{n-1} \oplus \mathcal{W}_{n-2} \oplus \cdots \oplus \mathcal{W}_{n-m} \oplus \mathcal{V}_{n-m}$$

Theorem: $\lim_{n\to\infty} \mathcal{V}_n = L^2(\mathbb{R})$

$$L^2(\mathbb{R}) = \bigoplus_{n=-\infty}^{\infty} \mathcal{W}_n = \mathcal{V}_m \oplus \left(\bigoplus_{n=m}^{\infty} \mathcal{W}_n \right)$$

Wavelets

$\mathcal{W}_n$ are wavelet spaces

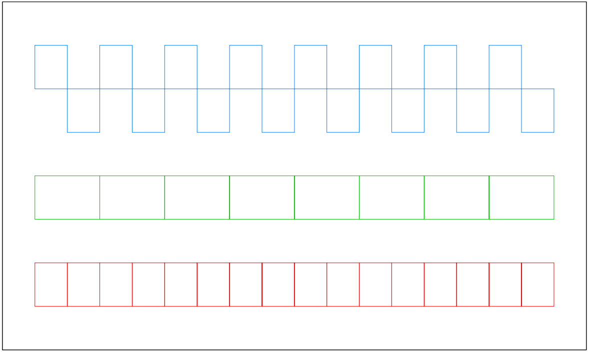
$$\psi(x) = D(\sum_{l=0}^{2K-1} (-)^l h_{2K-l-1} T^l \phi(x))$$

$\psi(x)$ is called the "Mother wavelet"

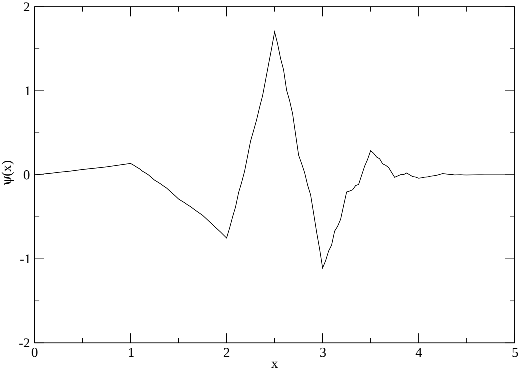
$$\psi_{nl}(x) := D^n T^l \psi(x)$$

$\{\psi_{nl}\}_l$ **orthonormal basis for $\mathcal{W}_n$**

Haar wavelets and scaling functions
(Daubechies K=1 wavelet and scaling function)



Daubechies' $K = 3$ mother wavelet



Scaling coefficients revisited

$$\textbf{support } [\psi(x)] = \textbf{support } [\phi(x)] = [0, 2K-1] \;.$$

h_l are determined up to space reflection by the requirements

$$(\psi, x^n) = 0, \quad n = 0, \cdots, K-1 \qquad (\phi, T^m\phi) = \delta_{m0}$$

$$\Downarrow$$

$$\sum_m m^n (-)^m h_{l-m} = 0 \qquad \sum_{l=0}^{2K-1} h_{l-2m} h_l = \delta_{m0}$$

$$\sum_{l=1}^{2k-1} h_l = \sqrt{2}$$

Properties

$\phi_m(x)$ **can locally pointwise represent polynomials of degree** $< K$**.**

$\psi_{mn}(x)$ **are orthogonal to polynomials of degree** $< K$**.**

$$L^2(\mathbb{R}) = \cdots \mathcal{W}_{n+2} \oplus \mathcal{W}_{n+1} \oplus \mathcal{W}_n \oplus \mathcal{V}_n$$

Wavelet Transform: W

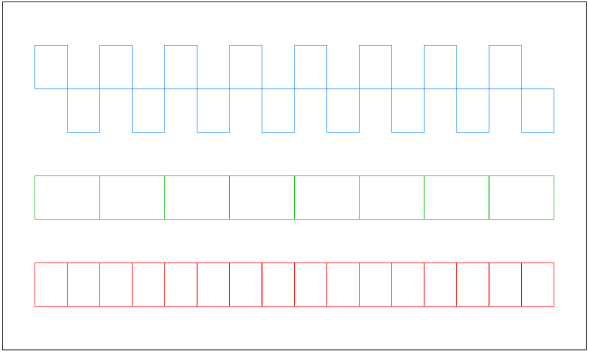
$$W : \mathcal{V}_n = \mathcal{W}_{n-1} \oplus \mathcal{W}_{n-2} \oplus \cdots \oplus \mathcal{W}_{n-m} \oplus \mathcal{V}_{n-m}$$

$$WW^t = I$$

W **fast** $O(N \times m)$

Computing W involves m applications of high and low-pass filters

Haar wavelets and scaling functions (Daubechies K=1 wavelet and scaling function)



JPEG?

$$\int \phi_{mn}(x)dx = \frac{1}{\sqrt{2^m}}$$

$$f(x) \approx \sum_n c_n \phi_{mn}(x)$$

$$c_n \approx \frac{1}{\sqrt{2^m}} f(n/2^m)$$

Wavelet transform W (orthogonal transformation)

$$f(x) = \sum_n c_m \phi_{mn}(x) = \sum_n a_n \phi_{m-l,n}(x) + \sum_n \sum_{k=1}^{l} b_{m-k,n} \psi_{m-k,n}(x)$$

Wavelet numerical analysis

Integrals involving functions that have simple scaling properties can be computed exactly from the renormalization group equation using linear algebra.

$$Dx^n = 2^{n+\frac{1}{2}} x^n$$

$$D\Theta(x) = \sqrt{2}\Theta(x)$$

$$D \ln x = \sqrt{2} \ln(x) + \sqrt{2} \ln(2)$$

$$D(\frac{df}{dx}) = 2 \frac{d}{dx}(Df)(x)$$

Strategy - use unitarity and the RG equation

$$(x^n, \phi) = (D^{-1} x^n, D^{-1} \phi)$$

Wavelet numerical analysis

Moments

$$\langle x^0 \rangle_\phi := (x^0, \phi) = 1$$

$$\langle x^m \rangle_\phi := (x^m, \phi) = (D^{-1} x^m, D^{-1}\phi) = 2^{-m-1/2} \sum_l h_l (x^m, T^l \phi) \quad \Rightarrow$$

$$\langle x^m \rangle_\phi = \frac{1}{\sqrt{2}} \frac{1}{2^m - 1} \sum_{l=0}^{2K-1} \sum_{n=0}^{m-1} h_l l^{m-n} \frac{m!}{n!(m-n)!} \langle x^n \rangle_\phi$$

recursion

$$\Downarrow$$

$\langle x^m \rangle_\phi, \quad \langle x^n \rangle_{\phi_{lm}}, \quad \langle x^n \rangle_{\psi_{lm}}$ **exact**

Integration

One-point quadrature

$$k > 1 \Rightarrow \langle x^n \rangle_\phi = \langle x \rangle_\phi^n, \qquad n = 0, 1, 2$$

$$\Downarrow$$

$$\int \phi(x) P(x) dx \approx P(\langle x^n \rangle_\phi)$$

exact for $P(x)$ a polynomial of degree 2!

$$\int \phi(x)(a + bx + cx^2) dx = a + b\langle x \rangle_\phi + c\langle x \rangle_\phi^2$$

Low-degree polynomials

$$x^m = \sum_n c_n^m \phi_n(x)$$

Expansion coefficients can be expressed in terms of moments.

exact for $m < K$

$$c_n^m = (T^n \phi, x^m) = \sum_{l=0}^{m} \frac{m! n^{m-l}}{l!(m-l)!} \langle x^l \rangle_\phi$$

$$c_n^0 = 1; \quad c_n^1 = n + \langle x^l \rangle_\phi, \quad \cdots$$

Smoothness, existence of derivatives

$$\phi(x) = \sqrt{2} \sum_{l=0}^{2K-1} h_l \phi(2x - l)$$

$$\Downarrow$$

$$\phi(n) = \sqrt{2} \sum_m H_{nm} \phi(m) \qquad H_{mn} := h_{2n-m}$$

$$\frac{d^l \phi}{dx^l}(n) = 2^l \sqrt{2} \sum_m H_{nm} \frac{d^l \phi}{dx^l}(m)$$

ϕ has l derivatives if H_{mn} has an eigenvalue $\lambda = 2^{-l-\frac{1}{2}}$

Computation of derivatives, differential equations

$$x = \sum_n c_n^1 \phi_n(x) \Rightarrow 1 = \sum_n c_n^1 \frac{d\phi_n}{dx}(x) \qquad c_n^1 = n + \langle x^1 \rangle_\phi$$

$$(\frac{d\phi}{dx})(x) = 2D^{-1} \sum_{l=0}^{2K-1} h_l (\frac{d\phi_l}{dx})(x).$$

$$\Downarrow$$

$$\frac{d\phi}{dx}(x) \qquad (\frac{d\phi}{dx}, \phi_n),$$

$$\frac{d\phi}{dx}(x) \approx \sum_n (\frac{d\phi}{dx}, \phi_n) \phi_n(x)$$

Derivatives

$$\frac{d\phi_{mn}}{dx}(x) \qquad \frac{d\psi_{mn}}{dx}(x)$$

can be calculated exactly at dyadic rationals, $(n/2^m)$.

$$(\phi_{mn}, \frac{d\phi_{mk}}{dx}) \qquad (\psi_{nl}, \frac{d\phi_{mk}}{dx}) \qquad (\phi_{mn}, \frac{d\psi_{mk}}{dx}) \qquad (\psi_{nl}, \frac{d\psi_{mk}}{dx})$$

can all be computed exactly.

Nonlinearities

$$\phi_{n_2}(x)\phi_{n_2}(x) \approx \sum_{n_3} \Gamma^{n_3}_{n_1,n_2}\phi_{n_3}(x)$$

$$\Gamma^{n_3}_{n_1,n_2} = \int \phi_{n_1}(x)\phi_{n_2}(x)\phi_{n_3}(x)dx =$$

$$\sum_{l_1,l_2,l_3} \sqrt{2}h_{l_1}h_{l_2}h_{l_3}\Gamma^{2n_3+l_3}_{2n_1+l_1,2n_2+l_2}$$

$$\sum_{n_3} \Gamma^{n_3}_{n_1,n_2} = \delta_{n_1,n_2} \qquad \Gamma^{n_3}_{n_1,n_2} = \Gamma^{0}_{n_1-n_3,n_2-n_3}$$

$$\Downarrow$$

$$\Gamma^{n_3}_{n_1,n_2}, \cdots$$

Boundary integrals

$$B_m = \int_m^\infty \phi(x)dx = \frac{1}{2}\sum_{l=1}^{2K-1} h_l B_{2m-l}$$

$$B_m = 0 \qquad m \geq 2K-1 \qquad B_m = 1 \qquad m \leq 0$$

$$B^m = \int_{-\infty}^m \phi(x)dx = 1 - B_m$$

Poles

$$J_n := (\phi_n, \frac{1}{x - i0^+}) = (D^{-1}\phi_n(x), D^{-1}\frac{1}{x - i0^+}) =$$

$$\sqrt{2}\sum_{l=0}^{2K-1} h_l(\phi_{2n+l}, \frac{1}{x}) = \sqrt{2}\sum_{l=0}^{2K-1} h_l J_{2n+l}$$

large n

$$J_n = \frac{1}{n}\sum_{m=0}^{\infty}\frac{\langle x^m\rangle_\phi}{n^m}$$

$$i\pi = \int_{-m}^{m}\frac{dx}{x - i0^+} = \sum_n J_n + \textbf{boundary terms}$$

Logarithmic singularities

$$L_n := (\phi_n, \ln) = (D^{-1}\phi_n, D(-1\ln) =$$

$$\frac{1}{\sqrt{2}} \sum_{l=0}^{2K-1} h_l L_{2n-l} - \ln(2)$$

large n

$$L_n = \ln(n) - \sum_{m=1}^{\infty} (-1)^m \frac{\langle x^m \rangle_\phi}{m n^m}$$

Moving singularities

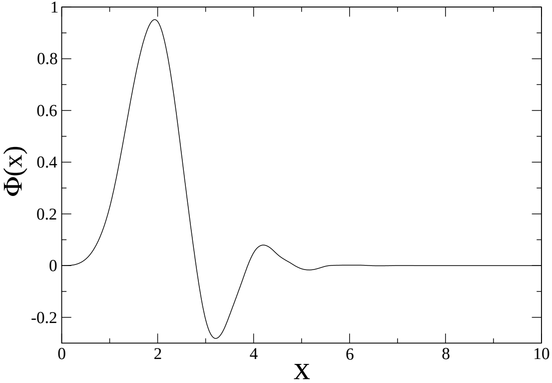
$$J_{mn} := \int \frac{\phi_m(x)\phi_n(y)}{k-x-y}dxdy =$$

$$\int \frac{\Phi(x)}{k-x-m-n}dx$$

$$\Phi(x) := \int \phi(y)\phi(x-y)dy$$

$$D\Phi(x) = \sum_l r_l T^l \Phi(x) \qquad \int \Phi(x)dx = 1$$

Autocorrelation function



Approximations and Wavelet Transforms

Fix resolution n of approximate solution

$$\mathcal{V}_n = \mathcal{W}_{n-1} \oplus \mathcal{W}_{n-2} \oplus \cdots \oplus \mathcal{W}_{n-m} \oplus \mathcal{V}_{n-m}$$

$$W : \mathcal{V}_n \to \mathcal{W}_{n-1} \oplus \mathcal{W}_{n-2} \oplus \cdots \oplus \mathcal{W}_{n-m} \oplus \mathcal{V}_{n-m}$$

Wavelet transform W (orthogonal transformation)

$$f(x) = \sum_m a_m \phi_{nm}(x) = \sum_m b_m \phi_{n-l,m}(x) + \sum_m \sum_{k=1}^{l} c_{n-k,m} \psi_{n-k,m}(x)$$

Sparse matrices

$$a_n \underbrace{\leftrightarrow}_{W} b_n, c_{nm}$$

b_{nm} small if $f(x)$ can be well approximated by degree $K-1$ polynomial on the support of $\psi_{nm}(x)$.

$$\textbf{supp}(\psi_{nm}(x)) \subseteq [2^{-n}m, 2^{-n}(m+2K-1)]$$

$$WMW^T = M_1 + M_2 \qquad \|M_2\| < \epsilon,\ M_1 \textbf{ sparse}$$

Application

$$f(x) = g(x) + \int \frac{K(x,y)}{y - i0^+} f(y) dy$$

$$f \approx \sum f_n \phi_{mn}(x)$$

$$f_l = g(x_l) + \sum_{nk} K(x_l, y_n) \Gamma^l_{kn} J_k f_n$$

$$\mathcal{W} \Rightarrow \text{solve} \Rightarrow \mathcal{W}^{-1} \Rightarrow$$

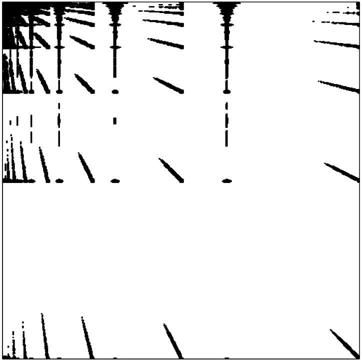
$$f(x) = g(x) + \sum_{nk} K(x, y_n) \Gamma^l_{kn} J_k f_n$$

$\psi_{mn}(x)$ **never had to be computed!**

$$\Gamma^l_{kn} = \int \phi_{ml}(x)\phi_{mk}(x)\phi_{mn}(x) dx \qquad J_n := 2^{m/2} \int dx \frac{\phi_{mn}(x)}{x - i0^+}$$

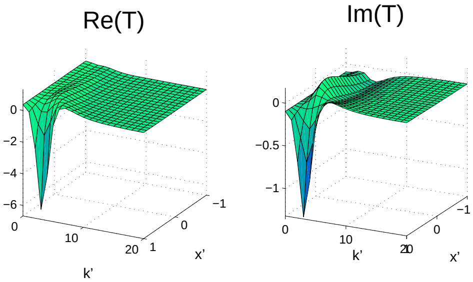
Required input

- **Kernel** $K(x, y)$ **at one-point quadrature points.**
- **Driving term** $B(x)$ **at one-point quadrature points.**
- **Integrals** Γ^{l}_{mn} **and** J_k **on one scale.**



K=3, E=10 MeV, m=7

ϵ	percent	on-shell value	on-shell error	mean-square error
0	100	-125.00480	0	0
10^{-9}	17.78	-125.00480	1.05×10^{-8}	2.56×10^{-8}
10^{-8}	11.38	-125.00480	5.14×10^{-8}	2.44×10^{-7}
10^{-7}	6.6	-125.00475	4.49×10^{-7}	1.88×10^{-6}
10^{-6}	3.76	-125.00269	1.69×10^{-5}	2.08×10^{-5}
10^{-5}	2.14	-124.99030	.000116	.000228
10^{-4}	1.24	-124.85112	.00123	.00217
10^{-3}	.72	-123.82508	.00944	.0117
10^{-2}	.38	-125.25766	.00202	.128



Re(T)
0
−2
−4
−6
0
10
20
k'
1
0
−1
x'
Im(T)
0
−0.5
−1
0
10
20
k'
1
0
−1
x'

Conclusion

- **Wavelet bases have all of the advantages of piecewise polynomial (spline) approximations with the additional properties:**
  - **orthonormal basis.**
  - **sparse matrix (even in momentum space).**
  - **wavelet transform automatically finds structure.**
  - **basis functions never need to be evaluated!**
- **Tested on two-body scattering using integral equations, differential equations, and time-dependent methods.**
- **Interesting possibilities for problems that couple many scales.**

Thanks!

Professor Hiller

U. S. D.O.E.