

Dynamical Implications of Cluster Properties in Poincaré Invariant Quantum Mechanics

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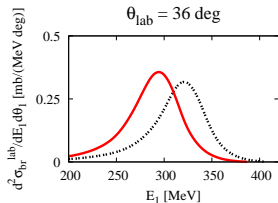
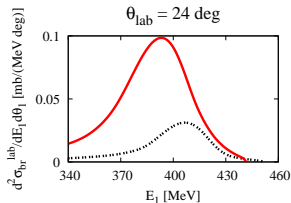
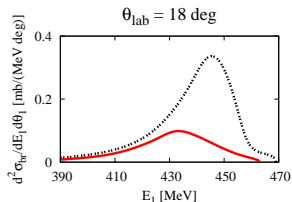
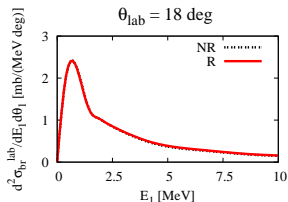


Background

- The three-body problem in Poincaré invariant quantum mechanics has been solved for laboratory beam energies up to 1 GeV (Elster, Glöckle, Golak, Kamada, Lin, Witała)
- The 4-body system is the simplest system where corrections due to cluster properties are observable.
- Cluster properties lead to new exchange currents in electron scattering from a three-nucleon system.

Calculations by C. Elster and T. Lin (unpublished)

$$E_{lab} = 495 \text{ MeV}$$



Outline

- Poincaré invariant quantum mechanics
- Cluster properties
- Group of scattering equivalences
- Cluster contribution to $N = 3$ exchange currents

Poincaré invariant quantum mechanics

Isolated particles

$$P = |\langle \Psi_f | \Psi_i \rangle|^2 = |\langle \Psi'_f | \Psi'_i \rangle|^2 = P'$$



$$U_1(\Lambda, a)$$



$$P_1^\mu, J_1^{\mu\nu}$$

(Wigner - 1939)

One-particle Hilbert space

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(M, J) irreducible representation

$$M, J, h_i; \Delta h_i$$

$$\mathcal{H}_1 = L^2(\sigma(h)) \quad |(m, j) \mathbf{h}\rangle$$

$$U_1(\Lambda, a)| (m, j) \mathbf{h}\rangle = \sum \int d\mathbf{h}' |(m, j) \mathbf{h}'\rangle D_{\mathbf{h}'\mathbf{h}}^{jm}(\Lambda, a)$$

$$D_{\mathbf{h}'\mathbf{h}}^{jm}(\Lambda, a) := \langle (m, j), \mathbf{h}' | U_1(\Lambda, a) | (m, j), \mathbf{h}\rangle$$

eg. $\mathbf{h} = (\vec{p}, \hat{z} \cdot \vec{j}_c)$

N-particle Hilbert space

$$\mathcal{H} = \bigotimes_{i=1}^N \mathcal{H}_i$$

Basis

$$|(m_1, j_1), \mathbf{h}_1, \dots, (m_N, j_N), \mathbf{h}_N\rangle$$

Kinematic Poincaré group

$$U_0(\Lambda, a) := \bigotimes U_i(\Lambda, a)$$

$$U_0(\Lambda, a) |(m_1, j_1), \mathbf{h}_1, \dots, (m_N, j_N), \mathbf{h}_N\rangle =$$

$$\sum \int |(m_1, j_1), \mathbf{h}'_1, \dots, (m_N, j_N), \mathbf{h}'_N\rangle \prod_{i=1}^n D_{\mathbf{h}'_i \mathbf{h}_i}^{j_i m_i}(\Lambda, a)$$

This is reducible!

N Free-particle irreducible representations

Poincaré Clebsch-Gordan coefficients

$$\bigotimes_{i=1}^N |(m_i, j_i), \mathbf{h}_i\rangle \langle CG \rangle = \langle CG \rangle \bigoplus_{j, m, \mathbf{d}} |(m, j), \mathbf{h}; \mathbf{d}\rangle$$

$$\bigotimes_{i=1}^n D_{\mathbf{h}'_i \mathbf{h}_i}^{j_i m_i}(\Lambda, a) \langle CG \rangle = \langle CG \rangle \bigoplus_{j, m, \mathbf{d}} D_{\mathbf{h}' \mathbf{h}}^{j m}(\Lambda, a)$$

↓

$$U_0(\Lambda, a) |(m, j), \mathbf{h}; \mathbf{d}\rangle = \sum \int |(m, j), \mathbf{h}' ; \mathbf{d}\rangle D_{\mathbf{h}' \mathbf{h}}^{j m}(\Lambda, a)$$

Dynamics

$$\langle (m', j'), \mathbf{h}'; \mathbf{d}' | V | (m, j), \mathbf{h}; \mathbf{d} \rangle = \delta_{jj'} \delta_{\mathbf{h}\mathbf{h}'} \langle m'; \mathbf{d}' | V^j | m; \mathbf{d} \rangle$$

Find simultaneous eigenstates, $|(\bar{m}, j), \mathbf{h}; \bar{\mathbf{d}}\rangle$, of

$$\bar{M} = M_0 + V, j^2, \mathbf{h}$$

$\bar{M}, j^2, \mathbf{h}, \Delta \mathbf{h}$ same commutation relations as $M_0, j^2, \mathbf{h}, \Delta \mathbf{h}$

$$|(\bar{m}, j), \mathbf{h}; \bar{\mathbf{d}}\rangle \quad (\text{complete})$$

$$\bar{U}(\Lambda, a) |(\bar{m}, j), \mathbf{h}; \bar{\mathbf{d}}\rangle = \sum \int d\mathbf{h}' |(\bar{m}, j), \mathbf{h}'; \bar{\mathbf{d}}\rangle D_{\mathbf{h}'\mathbf{h}}^{j\bar{m}}(\Lambda, a)$$

Cluster properties

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Poincaré invariance of isolated subsystems

Formulation of cluster properties

\mathbf{A}_i = subsystem \mathbf{S} = system

$$T_{\mathbf{A}_i}(x_i) := U_{\mathbf{A}_i}(I, x_i)$$

$$\mathbf{S} = \mathbf{A}_1 \cup \cdots \cup \mathbf{A}_m$$

$$\mathcal{H} = \mathcal{H}_{\mathbf{A}_1} \otimes \cdots \otimes \mathcal{H}_{\mathbf{A}_m}$$

$$\| [U(\Lambda, a) - \bigotimes_{i=1}^m U_{\mathbf{A}_i}(\Lambda, a)] \bigotimes_{i=1}^m T_{\mathbf{A}_i}(x_i) |\psi\rangle \| \rightarrow 0$$

as $(x_i - x_j)^2 \rightarrow +\infty$

Failure of cluster properties (2 + 1 system)

$$\mathbf{h} = (\vec{p}, j_{cz}) \quad \vec{p}_3 = \vec{q} + \vec{P}F(m_{12}, \vec{q}, \vec{P})$$

$$\langle \vec{P}, \vec{q}, m_{12} | V | \vec{P}', \vec{q}', m'_{12} \rangle = \\ \delta(\vec{P} - \vec{P}') \delta(\vec{q} - \vec{q}') \langle m_{12} | v(\vec{P}, \vec{q}) | m'_{12} \rangle$$

$$\langle \vec{P}, \vec{q}, m_{12} | e^{i\vec{p}_3 \cdot \vec{a}} V e^{-i\vec{p}_3 \cdot \vec{a}} | \vec{P}', \vec{q}', m'_{12} \rangle = \\ \delta(\vec{P} - \vec{P}') \delta(\vec{q} - \vec{q}') \underbrace{e^{i\vec{P} \cdot \vec{a} [F(m_{12}, \vec{q}, \vec{P}) - F(m'_{12}, \vec{q}, \vec{P})]}}_{\text{red bracket}} \langle m_{12} | v(\vec{P}, \vec{q}) | m'_{12} \rangle$$

$\rightarrow 0$ as $|\vec{a}| \rightarrow \infty$ for $\vec{P} \neq \vec{0}$.

Restoring cluster properties

$$U_{(12)(3)}(\Lambda, a) = \bar{U}_{(12)} \otimes U_3(\Lambda, a)$$

(Satisfies cluster properties)



$$\bar{U}_{(12)(3)}(\Lambda, a)$$

($\bar{M}_{(12)(3)}$ Commutes with three-body $j^2, \mathbf{h}, \Delta\mathbf{h}$)

How are these representations related?

Interactions (2 + 1) system

$$V := M(m_{12} + v) - M(m_{12})$$

$$\langle \vec{P}, \vec{q}, \vec{k} | v | \vec{P}', \vec{q}', \vec{k}' \rangle = \delta(\vec{P} - \vec{P}') \delta(\vec{q} - \vec{q}') \langle \vec{k} | v | \vec{k}' \rangle$$

$$\langle \vec{P}, \vec{p}_3, \vec{k} | v | \vec{P}', \vec{p}_3', \vec{k}' \rangle = \delta(\vec{P} - \vec{P}') \delta(\vec{p}_3 - \vec{p}_3') \langle \vec{k} | v | \vec{k}' \rangle$$

$$(m_{12} = \sqrt{m_1^2 + k^2} + \sqrt{m_2^2 + k^2} \quad m_{12} \iff k^2)$$

\Downarrow

$$\langle \vec{P}, \vec{q}, \vec{k} | \bar{S} | \vec{P}', \vec{q}', \vec{k}' \rangle = \delta(\vec{P} - \vec{P}') \delta(\vec{q} - \vec{q}') \langle \vec{k} | s | \vec{k}' \rangle$$

$$\langle \vec{P}, \vec{p}_3, \vec{k} | S | \vec{P}', \vec{p}_3', \vec{k}' \rangle = \delta(\vec{P} - \vec{P}') \delta(\vec{p}_3 - \vec{p}_3') \langle \vec{k} | s | \vec{k}' \rangle$$

$$k^2 = k'^2$$

 \Downarrow

$$\delta(\vec{P} - \vec{P}') \delta(\vec{p}_3 - \vec{p}_3') = \delta(\vec{P} - \vec{P}') \delta(\vec{q} - \vec{q}')$$

 \Downarrow

$$\bar{S} = S$$

Ekstein's Theorem

$$WW^\dagger = I \quad WMW^\dagger = \bar{M} \quad M_0 = \bar{M}_0$$

$$S = \bar{S} \Leftrightarrow \lim_{t \rightarrow \pm\infty} \|(W^\dagger - I)e^{-iM_0 t}|\psi\rangle\| = 0$$

$$W = \bar{\Omega}_\pm(\bar{M}, M_0)\Omega_\pm^\dagger(M, M_0)$$

(Ekstein 1960)

Dynamics related by W_p

$$S_p = \bar{S}_p \quad (p := (ij)(k))$$

⇓

$$W_p^\dagger \bar{M}_p W_p = M_p \quad W_p = \bar{\Omega}_{p\pm} \Omega_{p\pm}^\dagger$$

⇓

$$U_p(\Lambda, a) \Omega_{p+} \bar{\Omega}_{p+}^\dagger = \Omega_{p+} U_f(\Lambda, a) \bar{\Omega}_{p+}^\dagger = \Omega_{p+} \bar{\Omega}_{p+}^\dagger \bar{U}_p(\Lambda, a)$$

⇓

$$U_{ij}(\Lambda, a) \otimes U_k(\Lambda, a) = U_p(\Lambda, a) = W_p^\dagger \bar{U}_p(\Lambda, a) W_p$$

Summary: 2 + 1 body problem

$$\begin{array}{ccc}
 U_{(12)_0}(\Lambda, a) \otimes U_3(\Lambda, a) & \xrightarrow{\langle AB|C \rangle_0} & U_{(12)_0(3)}(\Lambda, a) \\
 \downarrow V_{(12)(3)} & & \downarrow \bar{V}_{(12)(3)} \\
 U_{(12)_i}(\Lambda, a) \otimes U_3(\Lambda, a) & \xrightarrow{\langle AB|C \rangle_i} & U_{(12)_i(3)}(\Lambda, a) \underbrace{\sim}_{W_p} \bar{U}_{(12)_i(3)}(\Lambda, a)
 \end{array}$$

There exist unitary transformations W that:

1. Preserve S .
2. Do not preserve cluster properties.
3. Do not modify the description of a free asymptotic particle.
4. Form a group, and are elements of a norm closed algebra of asymptotic constants.
- 5.

$$\lim_{|\vec{a}_1 - \vec{a}_2| \rightarrow \infty} \|(W_{12} - I)e^{i\vec{p}_1 \cdot \vec{a}_1 + i\vec{p}_2 \cdot \vec{a}_2} |\psi\rangle\| = 0$$

These operators can be used to restore cluster properties in the interacting three-body system!

Symmetric products

(Sokolov 1977)

$$W_p = \Omega_{\pm}(\bar{M}_p, M_0)\Omega_{\pm}^{\dagger}(M_p, M_0)$$

$$w_p := \ln(W_p)$$

$$w = \sum_p w_p \quad p \in \{(12)(3), (23)(1), (31)(2)\}$$

$$W = e^w$$

$$W \rightarrow W_p$$

W is a scattering equivalence!

Construction of $U(\Lambda, a)$

$$\begin{aligned}
 \bar{U}_{p_1}(\Lambda, a) \otimes U_{p_2}(\Lambda, a) &\xrightarrow{\underbrace{\quad}_{W_p}} \bar{U}_p(\Lambda, a) \longrightarrow \left\{ \begin{array}{c} \bar{M}_p \\ j \\ \mathbf{h} \end{array} \right\} \\
 &\longrightarrow \left\{ \begin{array}{c} \bar{M} = \sum_p \bar{M}_p - 2M_0 \\ j \\ \mathbf{h} \end{array} \right\} \longrightarrow \bar{U}(\Lambda, a) \\
 &\xrightarrow{\underbrace{\quad}_{W^\dagger}} U(\Lambda, a) = W^\dagger \bar{U}(\Lambda, a) W
 \end{aligned}$$

$$\bar{U}(\Lambda, a) \rightarrow \bar{U}_p(\Lambda, a) = W_p(\otimes_i U_{p_i}(\Lambda, a)) W_p^\dagger \quad W \rightarrow W_p$$

$$U(\Lambda, a) = W^\dagger \bar{U}(\Lambda, a) W \rightarrow W_p^\dagger W_p U_p(\Lambda, a) W_p^\dagger W_p = U_p(\Lambda, a)$$

Practical three-body calculation

1. Diagonalize (Coester 1965)

$$\bar{M} := \bar{M}_1 + \bar{M}_2 + \bar{M}_3 - 2M_0, j^2, \mathbf{h}$$

$$\bar{M}|(j, \bar{m})\mathbf{h}, \mathbf{d}\rangle = \bar{m}|(j, \bar{m})\mathbf{h}, \mathbf{d}\rangle$$

2. Transform eigenstates

$$|(j, m)\mathbf{h}, \mathbf{d}\rangle = W^\dagger |(j, \bar{m})\mathbf{h}, \mathbf{d}\rangle$$

3. Transformation W is a scattering equivalence. Cannot change S matrix elements or bound state properties.

4. Observable consequences for $N \geq 4$.

Currents

$$\bar{U}(\Lambda, a) = WU(\Lambda, a)W^\dagger$$

$$\bar{U}^\dagger(\Lambda, a)\bar{J}^\mu(0)\bar{U}(\Lambda, a) = \Lambda^\mu{}_\nu\bar{J}^\nu(0) \quad [\bar{P}_\mu, \bar{J}^\mu(0)] = 0$$

$$U^\dagger(\Lambda, a)J^\mu(0)U(\Lambda, a) = \Lambda^\mu{}_\nu J^\nu(0) \quad [P_\mu, J^\mu(0)] = 0$$

“Impulse approximations” natural in representations that cluster



$$\bar{J}^\mu(0) = WJ^\mu(0)W^\dagger$$

Summary

- Three-body bound state and scattering calculations based on Poincaré invariant quantum mechanics are possible (Elster, Glöckle, Golak, Kamada, Lin, Witała)
- Four-body systems are the simplest systems where corrections due to cluster properties can be seen in the scattering matrix.
- The construction of the exchange currents generated by restoring cluster properties was outlined.
- The importance of these corrections can now be tested. The required new input is W