

**Treatment of scattering integral  
equations with moving singularities using  
wavelet bases**

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**Research performed at the University of  
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**Supported by the U.S. D.O.E. Office of  
Science**

## Why use a wavelet basis?

- Used for data compression

JPEG 2000

FBI fingerprint archive

- Local orthonormal basis
- Automatically identify local structures
- Pointwise local representation of low degree polynomials
- Locally orthogonal to low degree polynomials
- $O(N)$  wavelet transform

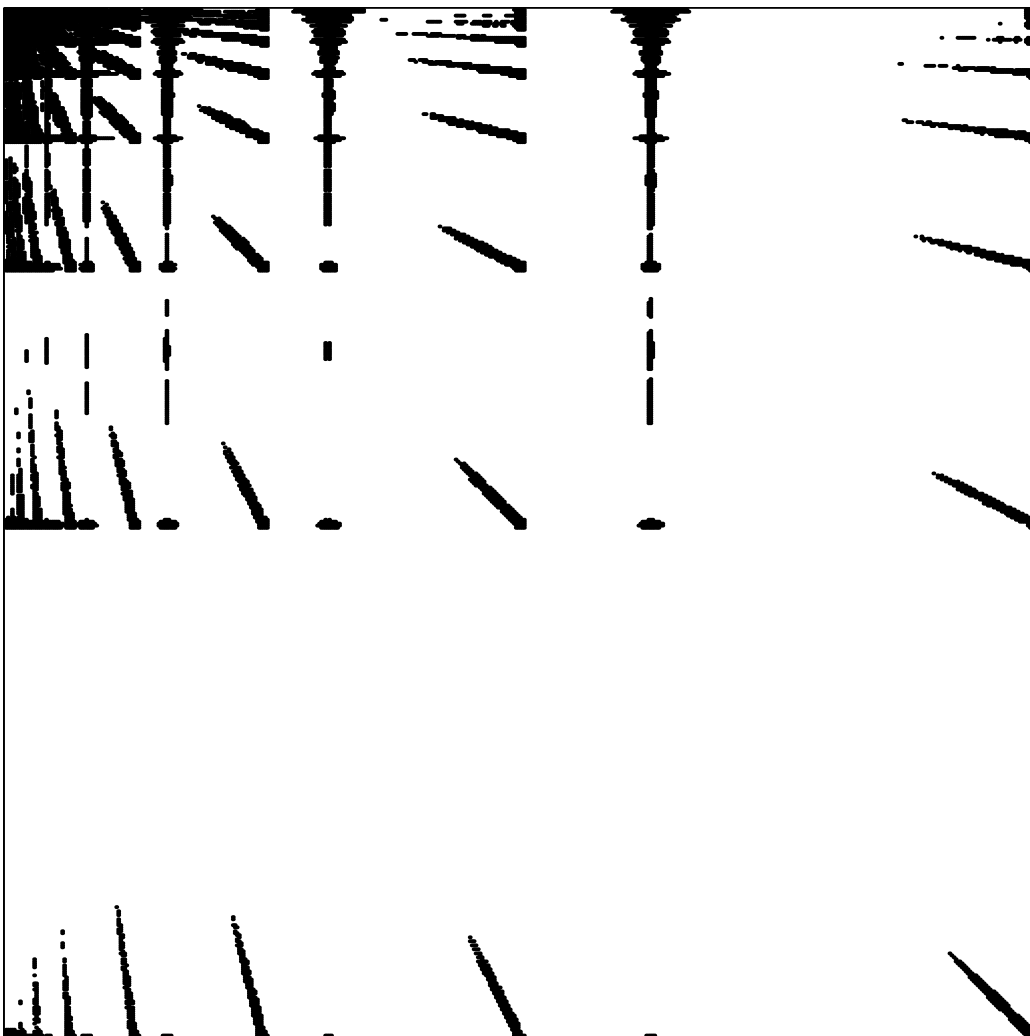
## Application to integral equations

- The wavelet transform gives accurate sparse-matrix approximations to (momentum-space) kernels of scattering integral equations.
- Integrals over the scattering cut can be computed efficiently and accurately.
- Accurate and efficient one-point quadrature rule.
- Clean separation of physical input and treatment of moving singularities.

## Summary of our work testing this method

- B. M. Kessler, G. L. Payne, W.N.P.  
Few-Body Systems, 33,1(2003)  
nucl-th/0211016.
- B. M. Kessler, G. L. Payne, W.N.P.  
Phys. Rev. C70,034003(2004)  
nucl-th/0406079.
- Fatih Bulut and W.N.P.  
nucl-th/0507051.

## Sparse momentum-space kernel for Malfliet-Tjon $V$ Interaction



## Scaling functions and wavelets

Scaling coefficients



Scaling equation & normalization condition



Multi-scale analysis



Wavelet basis, scaling basis, wavelet transform

## Elements of wavelet numerical analysis

Scaling equation & normalization condition



Moments, partial moments



Integrals over singular functions with scaling properties

Quadrature rules

Wavelet transform

Sloan interpolation

Basis function computation

## Scaling Coefficients

$$\sum_{l=0}^{2K-1} h_l = \sqrt{2}$$

$$\sum_{l=0}^{2K-1} h_l h_{l-2m} = \delta_{m0}$$

**Daubechies'  $K = 3$  scaling coefficients**

$h_l$	<b>K=3</b>
$h_0$	$(1 + \sqrt{10} + \sqrt{5 + 2\sqrt{10}})/16\sqrt{2}$
$h_1$	$(5 + \sqrt{10} + 3\sqrt{5 + 2\sqrt{10}})/16\sqrt{2}$
$h_2$	$(10 - 2\sqrt{10} + 2\sqrt{5 + 2\sqrt{10}})/16\sqrt{2}$
$h_3$	$(10 - 2\sqrt{10} - 2\sqrt{5 + 2\sqrt{10}})/16\sqrt{2}$
$h_4$	$(5 + \sqrt{10} - 3\sqrt{5 + 2\sqrt{10}})/16\sqrt{2}$
$h_5$	$(1 + \sqrt{10} - \sqrt{5 + 2\sqrt{10}})/16\sqrt{2}$

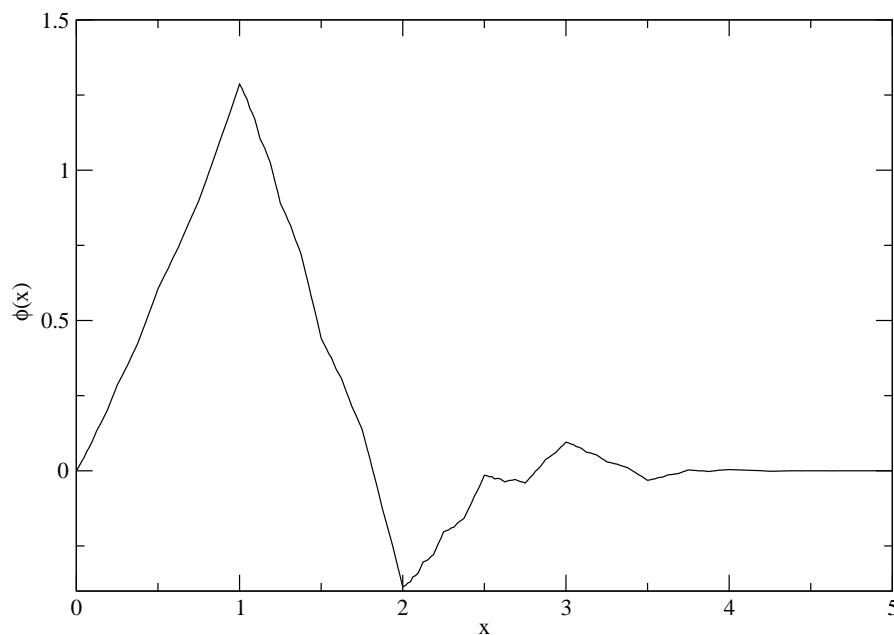
## Daubechies' scaling functions: $\phi(x)$

$$D\phi(x) = \sum_{l=0}^{2K-1} h_l T^l \phi(x) \quad \int dx \phi(x) = 1$$

$$Df(x) := \frac{1}{\sqrt{2}} f\left(\frac{x}{2}\right) \quad Tf(x) := f(x-1)$$

$$\text{supp}(\phi) = [0, 2K - 1]$$

Daubechies' K=3 scaling function



## Multiresolution analysis

$$\phi_{km}(x) = D^k T^m \phi(x) \quad (\phi_{km}, \phi_{kn}) = \delta_{mn}$$

**Resolution  $k$  approximation space:**

$$\mathcal{V}_k := \left\{ f(x) = \sum_m c_m \phi_{km}(x) \mid \sum_m |c_m|^2 < \infty \right\}$$

$$\cdots \mathcal{V}_{k-1} \supset \mathcal{V}_k \supset \mathcal{V}_{k+1} \supset \cdots$$

$$\lim_{k \rightarrow -\infty} \mathcal{V}_k = L^2(\mathbb{R}) \quad \lim_{k \rightarrow \infty} \mathcal{V}_k = \{\emptyset\}$$

$$\mathcal{V}_k \supset \mathcal{V}_{k+1} \quad \mathcal{V}_k = \mathcal{V}_{k+1} \oplus \mathcal{W}_{k+1}$$

$$\boxed{\mathcal{V}_k = \mathcal{W}_{k+1} \oplus \cdots \mathcal{W}_{k+m} \oplus \mathcal{V}_{k+m}}$$

$$\mathcal{W}_k := \left\{ f(x) = \sum_m c_m \psi_{km}(x) \mid \sum_m |c_m|^2 < \infty \right\}$$

**Wavelet Transform ( $O(N)$ ):**

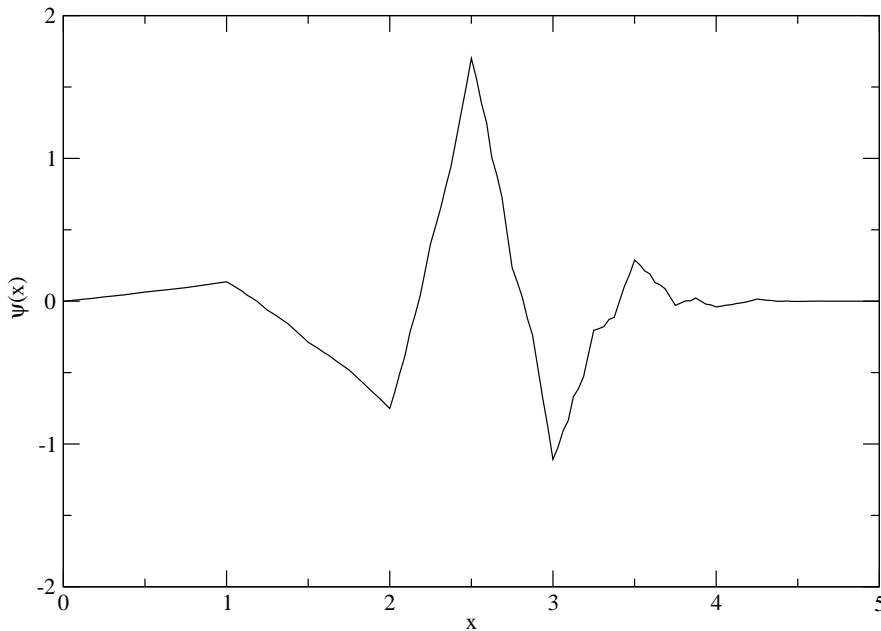
$$\mathcal{V}_k \Leftrightarrow \mathcal{W}_{k+1} \oplus \cdots \mathcal{W}_{k+m} \oplus \mathcal{V}_{k+m}$$

## Wavelets

$$\psi(x) := \sum_{l=0}^{2K-1} g_l D^{-1} T^l \phi(x) \quad g_l = (-)^l h_{2K-1-l}$$

$$\psi_{km}(x) = D^k T^m \psi(x)$$

$$(\psi_{km}, \psi_{ln}) = \delta_{kl} \delta_{mn} \quad (\phi_{km}, \psi_{ln}) = 0 \quad l \leq k$$



## Relativistic three-body scattering equations

$$F(k, q) = D(k, q) + \int \frac{R(k, q; k', q') dk' dq'}{z - \omega_1(q') - \omega_2(k', q') + i0^+} F(k', q')$$

$$\omega_1 = \sqrt{m^2 + q^2}$$

$$\omega_2 = \sqrt{4m^2 + 4k^2 + q^2}$$

↓

$$\bar{F}(x, y) = \bar{D}(x, y) + \int \frac{\bar{R}(x, y; x', y') dx' dy'}{z' - x' - y' + i0^+} \bar{F}(x', y')$$

**Approximations: Project on approximation space of resolution  $k$**

$$P_k f(x, y) = \sum_{mn} f_{mn}^k \phi_{km}(x) \phi_{kn}(y)$$

$$f_{mn}^k \approx 2^k f(2^k(\langle x \rangle + m), 2^k(\langle x \rangle + n))$$

$$\langle x \rangle = \frac{1}{\sqrt{2}} \sum_{l=0}^{2K-1} lh_l$$

$$\bar{F} \approx \bar{F}_k := P_k \bar{F}$$

$$\bar{D} \approx \bar{D}_k := P_k \bar{D}$$

$$\bar{R} \approx \bar{R}_k := P_k \bar{R} P_k$$

$$\bar{R}_k \bar{F}_k \approx \bar{R}_k \bar{F}_k P_k$$

$x$	$x^4$	$\sum x_m^2 x_n^2 I_{mnk} \phi_k$
-1.000000e + 01	1.000000e + 04	1.000000e + 04
-9.000000e + 00	6.561000e + 03	6.561000e + 03
-8.000000e + 00	4.096000e + 03	4.096000e + 03
-7.000000e + 00	2.401000e + 03	2.401000e + 03
-6.000000e + 00	1.296000e + 03	1.296000e + 03
-5.000000e + 00	6.250000e + 02	6.250002e + 02
-4.000000e + 00	2.560000e + 02	2.560002e + 02
-3.000000e + 00	8.100000e + 01	8.100015e + 01
-2.000000e + 00	1.600000e + 01	1.600010e + 01
-1.500000e + 00	5.062500e + 00	5.062574e + 00
-1.000000e + 00	1.000000e + 00	1.000050e + 00
-5.000000e - 01	6.250000e - 02	6.252593e - 02
-3.750000e - 01	1.977539e - 02	1.979528e - 02
-2.500000e - 01	3.906250e - 03	3.920094e - 03
-1.250000e - 01	2.441406e - 04	2.519414e - 04
0.000000e + 00	0.000000e + 00	1.757732e - 06
1.250000e - 01	2.441406e - 04	2.398553e - 04
2.500000e - 01	3.906250e - 03	3.895922e - 03
3.750000e - 01	1.977539e - 02	1.975902e - 02
5.000000e - 01	6.250000e - 02	6.247759e - 02
1.000000e + 00	1.000000e + 00	9.999534e - 01
2.000000e + 00	1.600000e + 01	1.599991e + 01
3.000000e + 00	8.100000e + 01	8.099986e + 01
4.000000e + 00	2.560000e + 02	2.559998e + 02
5.000000e + 00	6.250000e + 02	6.249998e + 02
6.000000e + 00	1.296000e + 03	1.296000e + 03
7.000000e + 00	2.401000e + 03	2.401000e + 03
8.000000e + 00	4.096000e + 03	4.096000e + 03
9.000000e + 00	6.561000e + 03	6.561000e + 03
1.000000e + 01	1.000000e + 04	1.000000e + 04

**Approximate equation - resolution  $k$   
scaling basis:**

$$\bar{F}_{mn} = \bar{D}_{mn} +$$

$$\sum \bar{R}_{m,n;m'n'} I_{m'm''m'''} I_{n'n''n'''} J_{m'',n''}(z) \bar{F}_{m''',n'''}$$

**where**

$$I_{lmn} := \int_0^\infty \phi_{kl}(x) \phi_{km}(x) \phi_{kn}(x) dx.$$

$$J_{m,n}(z) := \int_0^\infty dx dy \frac{\phi_{km}(x) \phi_{kn}(y)}{z - x - y + i0^+}.$$

**The scaling equation can be used to compute both of these integrals.**

**For the unrestricted integrals ( $[-\infty, \infty]$ ) the scaling equation ( $k = 0$ ) implies**

$$I_{lmn} = I_{0,m-l,n-l} := I_{m-l,n-l}$$

$$I_{mn} \neq 0 \quad \text{for} \quad m, n \in [-2K + 2, 2K - 2]$$

$$I_{mn} = \sqrt{2} \sum_{l_m l_n l_k=0}^{2K-1} h_{l_m} h_{l_n} h_{l_k} I_{2m+l_m-l_k, 2n+l_n-l_k}$$

$$\sum_{m=-2K+2}^{2K-2} I_{mn} = \delta_{n0}$$

**For  $K = 3$  this is a system of 81 linear equations that can be solved for the  $I_{mn}$ . These can be used directly to calculate all of the overlap integrals in the dynamical equation.**

For the unrestricted integrals ( $k = 0$ )

$$J_{mn}(l) = \int \frac{\phi(x)\phi(y)dx dy}{l - m - n - x - y + i0^+} = J_{l-m-n}$$

$$J_n = \frac{1}{n} \sum_{m=0}^{\infty} \sum_{l=0}^m \frac{1}{n^m} \frac{m!}{l!(m-l)!} \langle x^l \rangle \langle x^{m-l} \rangle$$

converges for  $|n| > 4K - 2$ . The scaling relation

$$J_n = \sum_{l'} h_l h_{l'} J_{2n-l-l'}$$

can be used to recurse up to  $n = -1$  and down to  $n = 4K - 1$ . This fixes  $J_n$  for regular values of  $n$

Another equation is needed to find the  $J_n$  at singular values of  $n$ . These equations fix the asymptotic condition

$$J_n = - \int \frac{\Phi(y)dy}{y - n + i0^+}$$

$$\Phi(y) := \int \phi(x - y)\phi(y)dy \quad \int \Phi(x)dx = 1$$

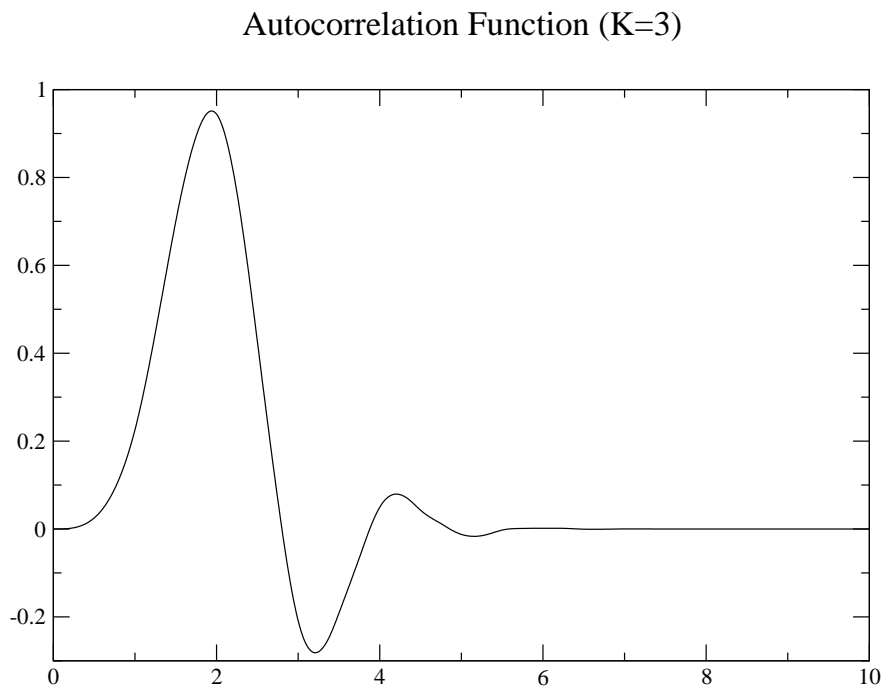
$$D\Phi = \sum_{l=0}^{4K-2} a_l T^l \Phi \quad \sum_n T^n \Phi = 1$$

$$i\pi = - \int_{-m}^m \frac{dx}{x - i0^+} = - \sum_n \int_{-m}^m dx \frac{\Phi(x_n)}{x - i0^+} =$$

$$\sum_{n=-m}^m J_n + \text{boundary terms}$$

The system can be solved for all of the singular  $J_n$ 's.

## Autocorrelation function $\Phi(x)$ :



**Table 5 -  $\bar{J}_n$** 

m	J
0	$-6.400535e - 01 + i0.000000e + 00$
1	$-1.570288e + 00 - i7.088321e - 01$
2	$6.615596e - 01 - i2.966393e + 00$
3	$1.719674e + 00 + i6.560028e - 01$
4	$6.721642e - 02 - i1.554882e - 01$
5	$3.595012e - 01 + i3.858342e - 02$
6	$2.261977e - 01 - i5.023224e - 03$
7	$1.853414e - 01 - i4.383938e - 04$
8	$1.569998e - 01 - i3.586652e - 06$
9	$1.357112e - 01 - i4.424016e - 09$
10	$1.195044e - 01 + i0.000000e + 00$

## Conclusions

Wavelet bases can be used to reduce two-variable momentum-space scattering integral equations to sparse-matrix equations with the following properties.

1. The smooth ( $\bar{R}_{mn;m'n'}$ ,  $\bar{D}_{mn}$ ,  $I_{lmn}$ ) and singular contributions ( $J(z)_{mn}$ ) to the integral equation are cleanly separated.
2. The singular contribution is independent of the physics input and can be computed efficiently to any desired precision using the scaling equation and linear algebra.
3. The physics input can be included using the one-point quadrature approximation.
4. The wavelet transform identifies the most important contributions to the equation up to a given minimal resolution.
5. The wavelet transform leads to equivalent sparse-matrix formulations of the equations that can be solved efficiently by iterative methods.