Relativity and Quantum Theory

W. Polyzou
University of Iowa

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Background

- Pre-1939 approaches to combining relativity and quantum theory were strongly influenced by appealing to properties of classical wave equations.
- Pure quantum interpretation was initiated by Wigner in 1939 .

- **Principle of Relativity:**
    - **There exist a class of preferred reference frames called inertial frames.**
    - **Experiments on isolated systems cannot make an absolute determination of an inertial frame.**

Classical Relativity

- Classical observables are solutions of differential equations.
- Symmetries of classical observables lead to symmetries of the differential equations.

$$\Updownarrow$$

- The Principle of Relativity can be applied alternatively to the equations or solutions.

Quantum Relativity

- **Quantum obesrvables are probabilities,**

$$P := |\langle\psi|\phi\rangle|^2$$

Probabilities are representation independent.

$$|\psi'\rangle = W|\psi\rangle \qquad P = |\langle\psi|\phi\rangle|^2 = |\langle\psi'|\phi'\rangle|^2 = P'$$

- **Equations for the vectors are representation dependent.**

$$(H - E)|\psi\rangle = 0 \qquad (H' - E)|\psi'\rangle = 0 \qquad H' = WHW^\dagger$$

- **Symmetries of the probabilities do not necessarily imply symmetries of the equations.**

Symmetry groups

- Inertial frames are related by a transformation group.
- The symmetry group of special relativity is the Poincaré group (Lorentz group + space-time translations).
- The symmetry group of classical relativity is the Galilean group.
- The weak interaction implies that the discrete transformations of space reflection and time reversal are not included in the symmetry group of relativity.

Wigner - Bargmann Theorem

$$X \underbrace{\longrightarrow}_{(\Lambda,a)} X'$$

$$\boxed{P' = |\langle\Psi'|\Phi'\rangle|^2 = |\langle\Psi|\Phi\rangle|^2 = P}$$

$$\boxed{|\Psi'\rangle = U(\Lambda, a)|\Psi\rangle}$$

$$U^\dagger(\Lambda, a)U(\Lambda, a) = I \qquad U(\Lambda_2, a_2)U(\Lambda_1, a_1) = U(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2)$$

Minimal elements of a Poincaré invariant quantum theory

- **Hilbert space $\mathcal{H}$ - includes vectors, a complete inner product, $(\cdot,\cdot)$,and a positive norm.**
- **Unitary representation, $U(\Lambda, a) : \mathcal{H} \to \mathcal{H}$ of the Poincaré group.**

Additional desirable properties

- **Stability: Energy bounded from below.**
- **Cluster Properties: Isolated subsystems are independently Poincaré invariant.**
- **Microlocality: Observables associated with any space-like separated regions commute (normally requires fields.)**

Challenge - Find non-trivial theories with these properties!

- Requirements are satisfied for free fields ⇒ requirements are internally consistent.
- Mathematically consistent theories with non-trivial interactions satisfying all of the requirements are not known! (4-dimensional space time - Clay Millineum prizes)
- Is it possible to formulate meaningful models by weakening some of the requirements?

I discuss four possibilities:

(All give up microscopic locality)

- **Poincaré covariant quantum mechanics**
- **Particle quantum mechanics**
- **Euclidean relativistic quantum mechanics**
- **Covariant constraint dynamics**

Remarks

- **The focus is on the construction of $\mathcal{H}$ and $U(\Lambda, a)$**
- **Difficulties with each approach will be contrasted.**

There is no free lunch!

Free spinless particles

Example 1: Covariant wave functions

$$\langle x|f\rangle \to \langle x|f'\rangle = \langle x|U(\Lambda,a)|f\rangle = \langle \Lambda x + a|f\rangle$$

$$\langle f|g\rangle = \langle f'|g'\rangle$$

$$= \int d^4x d^4y f^*(x) W(x,y) g(y)$$

$$W(x,y) \to W(x-y) = W(\Lambda(x-y))$$

$$W_{ij} := W(x_i,x_j) \geq 0 \qquad \{x_i\}_{i=1}^N$$

Special case: Scalar quantum field theory

$$|f\rangle = \int f(x)\phi(x)|0\rangle dx$$

$$\langle f|g\rangle = \int d^4xd^4yf^*(x)\langle 0|\phi(x)\phi(y)|0\rangle g(y)$$

$$W(x,y) = \langle 0|\phi(x)\phi(y)|0\rangle$$

Lehmann representation of the 2 point Wightman function

$$W(x,y) = const \int e^{ip\cdot(x-y)}\delta(p^2+m^2)\theta(p^0)\rho(m)d^4pdm$$

Example 2: Particle quantum mechanics

$$\langle f|g\rangle = \int d^4p f^*(p)\theta(p_0)\delta(p^2+m^2)g(p) =$$

$$\int \frac{f^*(\omega(\mathbf{p}),\mathbf{p})}{\sqrt{2\omega(\mathbf{p})}} d\mathbf{p} \frac{g(\omega(\mathbf{p}),\mathbf{p})}{\sqrt{2\omega(\mathbf{p})}}$$

$$\langle \mathbf{p}|f_p\rangle = \frac{f_{cov}(\omega(\mathbf{p}),\mathbf{p})}{\sqrt{2\omega(\mathbf{p})}}$$

$$U(\Lambda, a)|\mathbf{p}\rangle = |\mathbf{p}'\rangle \sqrt{\frac{\omega(\mathbf{p}')}{\omega(\mathbf{p})}} e^{i\Lambda p\cdot a}$$

Example 3: Covariant constraint dynamics

$$C(p) := \theta(p_0)\delta(p^2 + m^2)$$

$$\langle f|g\rangle = \int f^*(p)C(p)g(p)dp$$

N Particles

$$\langle f|g\rangle = \int f^*(p_1, \cdots, p_N)\prod_i C(p_i)g(p_1, \cdots, p_N)dp^{4N}$$

$[C(p_i), C(p_j)] \approx 0$ **(first class constraints)**

Example 4: Euclidean relativistic quantum mechanics

$$\langle \tau, \mathbf{x} | f \rangle = 0 \qquad \tau \leq 0 \qquad \theta(\tau, \mathbf{x}) := (-\tau, \mathbf{x})$$

$$\langle f | g \rangle_m = (f, \Pi^+ \theta G_e \Pi^+ g)_e =$$

$$:= \int f(\theta(x)) \langle f | \tau_x, \mathbf{x} \rangle \frac{e^{-ip\cdot(x-y) d^4 p}}{p^2 + m^2} \langle \tau_y, \mathbf{y} | g \rangle d^4 x d^4 y = \int \hat{f}^*(\mathbf{p}) \hat{g}(\mathbf{p}) d\mathbf{p}$$

where

$$\hat{f}(\mathbf{p}) = \frac{\sqrt{\pi} f(x^0, \mathbf{x}) e^{-\omega(\mathbf{p}) x^0}}{\sqrt{\omega(\mathbf{p})}} e^{i\mathbf{p}\cdot\mathbf{x}} d^4 x$$

$$H = H^\dagger = \frac{\partial}{\partial \tau}$$

$$\mathbf{K} = \mathbf{K}^\dagger = -i(\mathbf{x} \frac{\partial}{\partial \tau} - \tau \frac{\partial}{\partial \mathbf{x}})$$

Spin - Elementary Building Blocks - $SL(2,C)$

Matrix representation of 4-vectors

$$P = p^\mu \sigma_\mu = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix} \qquad p^\mu = \frac{1}{2}\mathbf{Tr}(\sigma_\mu P)$$

$$\mathbf{det}(P) = (p^0)^2 - \mathbf{p}^2 \qquad P = P^\dagger$$

Lorentz transformations

$$P' = APA^\dagger \qquad \mathbf{det}(A) = 1 \qquad A \in SL(2,C)$$

preserves

$$\mathbf{det}(P) = \mathbf{det}(P') \qquad P' = P'^\dagger$$

Relation to $SO((1,3)$

$$\Lambda^{\mu}{}_{\nu} = \frac{1}{2}\mathbf{Tr}(\sigma^{\mu} A \sigma^{\nu} A^{\dagger}) \qquad \Lambda^{\mu}{}_{\nu} \Leftrightarrow A, -A$$

Space reflection

$$\sigma_2 P^* \sigma_2 = p^0 \sigma_0 + (-\mathbf{p}) \cdot \boldsymbol{\sigma} \qquad \sigma_2 A^* \sigma_2 = (A^{\dagger})^{-1}$$

$$P' = APA^{\dagger} \qquad \sigma_2 P^{*\prime} \sigma_2 = (A^{\dagger})^{-1} \sigma_2 P^* \sigma_2 A^{-1}$$

Space reflected representation is inequivalent.

$$(A^{\dagger})^{-1} \neq SAS^{-1}$$

$$A = \pm exp(\mathbf{z} \cdot \boldsymbol{\sigma}) \qquad \mathbf{det}(A) = exp(\mathbf{z} \cdot \mathbf{Tr}(\boldsymbol{\sigma})) = 1$$

Rotationless Lorentz transformations -

$$A = B(\mathbf{p}/m) = B(\mathbf{p}/m)^\dagger > 0$$

$$B(\mathbf{p}/m) = \pm exp(\frac{\boldsymbol{\rho}}{2} \cdot \boldsymbol{\sigma}) \qquad \mathbf{p}/m = \hat{\boldsymbol{\rho}} \sinh(\rho)$$

$$\sqrt{P} = B(\mathbf{p}/m)$$

Rotations - $SU(2)$

$$A = U(\boldsymbol{\theta}) = \pm exp(i\frac{\boldsymbol{\theta}}{2} \cdot \boldsymbol{\sigma})$$

Spinors (Lorentz)

$$A := A^{a}{}_{b} \qquad A^{*} := A_{\dot a}{}^{\dot b}$$

$$\sigma_2 A^{*} \sigma_2 = (A^{-1})^{\dagger} := A^{\dot a}{}_{\dot b} \qquad \sigma_2 A \sigma_2 = (A^{-1})^{T} := A^{a}{}_{b}$$

$$\xi^{a} \longrightarrow (\xi')^{a} = A^{a}{}_{b}\xi^{b} \qquad \xi^{\dot a} \longrightarrow (\xi')^{\dot a} = A^{\dot a}{}_{\dot b}\xi^{\dot b}$$

$$\xi_{a} \longrightarrow (\xi')_{a} = A_{a}{}^{b}\xi_{b} \qquad \xi_{\dot a} \longrightarrow (\xi')_{\dot a} = A_{\dot a}{}^{\dot b}\xi_{\dot b}$$

$$\xi^{a}\xi_{a} = (\xi')^{a}(\xi')_{a} \qquad \xi^{\dot a}\xi_{\dot a} = (\xi')^{\dot a}(\xi')_{\dot a}$$

$\sigma_2 =$ **metric spinor**

The spinor scalar products are invariant but not positive

Spinors (Dirac (ξ) vs Poincaré (χ))

$$\xi = \begin{pmatrix} \xi^{a} \\ \xi^{\dot{a}} \end{pmatrix}$$

$$\xi \rightarrow \xi' = \xi \begin{pmatrix} A & 0 \\ 0 & (A^{\dagger})^{-1} \end{pmatrix}$$

$$\chi(p) = \begin{pmatrix} \xi^{a} B(\mathbf{p}/m)_{a}{}^{b} \\ \xi^{\dot{a}} B(\mathbf{p}/m)_{\dot{a}}{}^{\dot{b}} \end{pmatrix}$$

$$\chi(p) \rightarrow \chi'(p') = \chi(\Lambda p) \begin{pmatrix} R_{w}(A,p) & 0 \\ 0 & R_{w}(A,p) \end{pmatrix}$$

$$R_{w}(A,p) = B^{-1}(\mathbf{p}'/m) A B(\mathbf{p}/m) = B(\mathbf{p}'/m)(A^{\dagger})^{-1} B^{-1}(\mathbf{p}/m)$$

$$(R^{\dagger})^{-1} = R$$

Modifications for spin:

$$\xi \longrightarrow \xi(p)$$

$$\xi^\dagger(p)P\xi(p) = \xi^\dagger(p')P'\xi(p') > 0 \qquad p^2 \quad \textbf{time-like}$$

$$P, P^*, \sigma_2 P \sigma_2, \sigma_2 P^* \sigma_2 > 0$$

$$P = B(\mathbf{p}/m)(m\sigma_0)B^\dagger(\mathbf{p}/m) = B(\mathbf{p}/m)B(\mathbf{p}/m)$$

Free particles with spin

Covariant quantum mechanics

$$U(\Lambda, a)|p, b\rangle = \sum_{b'} |\Lambda p, b'\rangle e^{i\Lambda p\cdot a} A_{b'b}$$

$$f(p, b) = \langle p, b|f\rangle$$

$$(f, g) = (f', g') = \int d^4 p d^4 q f^*(p, a) W_{ab}(p, q) g(q, b)$$

$$W_{ab}(p, q) = \delta^4(p - q)\delta(p^2 + m^2)\theta(p^0)P_{ab}$$

$$P_{ab} = (B(\mathbf{p}/m)B(\mathbf{p}/m))_{ab}$$

(right handed - can also treat conjugate representation)

Example: Quantum field theory - spin $1/2$

$$|f\rangle = \int f_a(p)\Psi_a^\dagger(p)|0\rangle d^4p$$

$$\langle f|g\rangle = \int d^4p d^4q f_a^*(p)\langle 0|\Psi_a(p)\Psi_b^\dagger(p)|0\rangle g_b(q)$$

$$W_{ab}(p,q) = 2m\delta^4(p-q)\delta(p^2+m^2)\theta(p^0)u_a(p)u_b^\dagger(p)$$

$$u_a(p) = \begin{pmatrix} B(\mathbf{p}/m) & 0 \\ 0 & B(\mathbf{p}/m)^{-1} \end{pmatrix}_{ab} u_b(\mathbf{0})$$

Lehmann representation of the 2 point Wightman function

Particle quantum mechanics

$$(f,g) = \int d^4p f_a^*(p)\theta(p_0)\delta(p^2+m^2)P_{ab}g_b(p) =$$

$$\int \frac{f^*(\omega(\mathbf{p}),\mathbf{p})}{\sqrt{2\omega(\mathbf{p})}} B(\mathbf{p}_m) d\mathbf{p} B(\mathbf{p}_m)\frac{g(\omega(\mathbf{p}),\mathbf{p})}{\sqrt{2\omega(\mathbf{p})}}$$

$$f_p(\mathbf{p}) = B(\mathbf{p}_m)\frac{f_{cov}(\omega(\mathbf{p}),\mathbf{p})}{\sqrt{2\omega(\mathbf{p})}}$$

$$U(\Lambda,a)|\mathbf{p},\alpha\rangle = |\mathbf{p}'\rangle\sqrt{\frac{\omega(\mathbf{p}')}{\omega(\mathbf{p})}}\, e^{i\Lambda p\cdot a}\underbrace{[B^{-1}(\mathbf{p}'{}_m)AB(\mathbf{p}_m)]}_{R_w(\Lambda,p)}$$

Same Wigner rotation for $B(\mathbf{p}/m)$ and $B^{-1}(\mathbf{p}/m)$

Covariant constraint dynamics

$$C(p) := \theta(p_0)\delta(p^2 + m^2)P_{ab}$$

$$(f, g) = \int f_a^*(p)C(p)P_{ab}g_b(p)dp$$

N Particles

$$(f, g) = \int f^*(p_1, \dot{a}_1 \cdots, p_N, \dot{a}_n) \prod_i C_{\dot{a}_i b_i}(p_i) g(p_1, b_1 \cdots, p_N, b_N)dp^{4N}$$

$$[C(p_i), C(p_j)] \approx 0$$

Euclidean relativistic quantum mechanics

$$f(x^0,\mathbf{x}) = 0 \qquad x^0 \leq 0 \qquad \theta((x^0,\mathbf{x})) := (-x^0,\mathbf{x})$$

$$(f,g) := \int f(\theta(x)) \frac{e^{-ip\cdot(x-y)d^4p}}{p^2+m^2} P_e g(y) d^4x d^4y =$$

$$\int \hat{f}^*(\mathbf{p})\hat{g}(\mathbf{p})d\mathbf{p}$$

where

$$\hat{f}(\mathbf{p}) = \frac{\sqrt{\pi} f(x^0,\mathbf{x}) e^{-\omega(\mathbf{p})x^0}}{\sqrt{\omega(\mathbf{p})}} B(\mathbf{p}/m) e^{i\mathbf{p}\cdot\mathbf{x}} d^4x$$

- Observations

- 2N component spinors can appear in the Lorentz covariant representations to treat inequivalent representations symmetrically (useful for linear realizations of parity and chiral transformations)
- In Poincaré covariant representations both right and left handed spinors have equivalent representation of Wigner rotations.
- Representations differ on whether the boost $B(\mathbf{p})$ is in the wave function or kernel of the scalar product.

Interactions

Covariant wave functions

$$W_4(x_1, x_2; y_1, y_2) =$$

$$W_2(x_1, y_1)W_2(x_2, y_2) \pm W_2(x_1, y_2)W_2(x_2, y_1) + W_T(x_1, x_2; y_1, y_2)$$

- W_4 **must be positive and covariant.**
- **Dynamics in** W_4. **No simple relation to a Lagrangian or Green's function.**

Interactions

Particle Quantum Mechanics

$$\mathcal{H} \to \mathcal{H}_1 \otimes \mathcal{H}_2$$

$$U(\Lambda, a) \to \{H, \mathbf{P}, \mathbf{J}, \mathbf{K}\}$$

$$H = \sum_i T_i + \sum_{i<j} V_{ij} + \cdots$$

$$\mathbf{K} = \sum_i \mathbf{K}_i + \sum_{i<j} \mathbf{K}_{ij} + \cdots$$

$$\vdots$$

- **Interactions can be added to mass Casimir operator in irreducible free particle representation.**

$$M_0 \rightarrow M = M_0 + V$$

- **Construction fails to satisfy cluster properties. Repairable for fixed number of particles.**

Interactions

Covariant constraint dynamics

$$C_{\pm}(p_1, p_2) = \delta(p_1^2 + m_1^2 \pm p_2^2 \pm m_2^2)\theta(p_1^0)\theta(p_1^0)$$

$$C_I = C_+ + \Phi_{ij} \qquad [\Phi_{ij}, C_-] \approx 0$$

$$C_{1I} = \frac{1}{2}(C_I + C_-) \qquad C_{2I} = \frac{1}{2}(C_I - C_-)$$

$$W = C_{1I} C_{2I}$$

- **Cluster properties are difficult to satisfy form more than two particles.**

$$C_i = C_{i0} + \sum_j C_{ij} + \sum_{j<k} C_{ijk} + \cdots$$

$$[C_i, C_j] \approx 0$$

- **This method has been used successfully in calculations.**

Interactions

Euclidean relativistic quantum mechanics

$$G_{e2} \to G_{e4} = G_{e2} G_{e2} + G_{e2} G_{e2} K_e G_{e4}$$

$$\langle f | g \rangle_m = (f, \Pi_{1+} \Pi_{2+} \theta_1 \theta_2 G_4 \Pi_{1+} \Pi_{2+} g)_e$$

- **Positivity of scalar product is not automatically preserved, even for a small kernel.**

- **Connection with Lagrangians and Euclidean Green functions.**

- **Analytic continuation is not needed.**

Summary

- All four methods give identical treatments of free particles.
- Relations are easy to understand for free particles.
- Natural approaches for adding interactions appear very different in each case.
- The Euclidean approach has the most straight forward relation to field theory phenomenology. It has special difficulties.
- The non-covariant approaches are the most developed.