Currents and Cluster Properties in Poincaré Invariant Quantum Mechanics

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12/5/06

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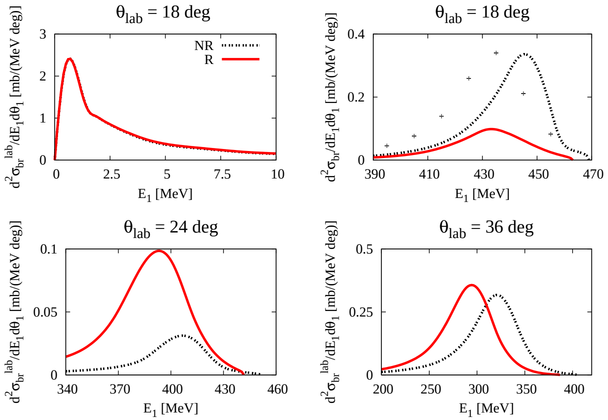
Background

- The three-body problem in Poincaré invariant quantum mechanics has been solved for laboratory beam energies up to 1 GeV (Elster, Glöckle, Golak, Kamada, Lin, Skibiński, Witała).
- The 4-body system is the simplest system where corrections due to cluster properties are observable.
- Cluster properties lead to new exchange currents in electron scattering from a three-nucleon system.

Calculations by C. Elster and T. Lin (unpublished)

$$E_{lab} = 495 \text{ MeV}$$



Experiment (p,n) charge exchange cross section: X. Y. Chen, et. al. PRC 47, 2159 (1993)

Outline

- **Poincaré invariant quantum mechanics**
- **Cluster properties**
- **Group of scattering equivalences**
- **Cluster contribution to $N = 3$ exchange currents**

Poincaré invariant quantum mechanics

- Change of inertial frame is a symmetry of a quantum theory
- Symmetries of quantum theories preserve probabilities
- Preserving expectation values and ensemble averages follow.
- Wave functions are not observable.

Poincaré invariant quantum mechanics

Isolated particles

$$P = |\langle \Psi_f | \Psi_i \rangle|^2 = |\langle \Psi'_f | \Psi'_i \rangle|^2 = P'$$

$$\Updownarrow$$

$$U_1(\Lambda, a)$$

$$\Updownarrow$$

$$P_1^{\mu}, J_1^{\mu\nu}$$

(Wigner - 1939)

One-particle Hilbert space

$$||$$

(M, J) irreducible representation

$$M, J, h_i, \Delta h_i$$

$$\mathcal{H}_1 = L^2(\sigma(h)) \qquad |(m,j)\,\mathbf{h}\rangle$$

$$U_1(\Lambda, a)|(m,j)\,\mathbf{h}\,\rangle = \sum \int d\mathbf{h}|(m,j)\,\mathbf{h}'\,\rangle D^{jm}_{\mathbf{h}'\mathbf{h}}(\Lambda, a)$$

$$D^{jm}_{\mathbf{h}'\mathbf{h}}(\Lambda, a) := \langle (m,j), \mathbf{h}'\,|U_1(\Lambda, a)|(m,j), \mathbf{h}\,\rangle$$

eg. $\quad \mathbf{h} = (\vec{p}, \hat{z} \cdot \vec{j}_c)$

N-particle Hilbert space

$$\mathcal{H} = \bigotimes_{i=1}^{N} \mathcal{H}_i$$

Basis

$$|(m_1, j_1), \mathbf{h}_1, \cdots (m_N, j_N), \mathbf{h}_N\rangle$$

Kinematic Poincaré group

$$U_0(\Lambda, a) := \bigotimes U_i(\Lambda, a)$$

$$U_0(\Lambda, a)|(m_1, j_1), \mathbf{h}_1, \cdots (m_N, j_N), \mathbf{h}_N\rangle =$$

$$\sum \int |(m_1, j_1), \mathbf{h}'_1, \cdots, (m_N, j_N), \mathbf{h}'_N\rangle \prod_{i=1}^{N} D^{j_i m_i}_{\mathbf{h}'_i \mathbf{h}_i}(\Lambda, a)$$

This is reducible!

N Free-particle irreducible representations

Poincaré Clebsch-Gordan coefficients

$$\bigotimes_{i=1}^{N} |(m_i, j_i), \mathbf{h}_i\rangle < CG >=< CG > \bigoplus_{j,m,\mathbf{d}} |(m, j), \mathbf{h}; \mathbf{d}\rangle$$

$$\bigotimes_{i=1}^{N} D^{j_i m_i}_{\mathbf{h}'_i \mathbf{h}_i}(\Lambda, a) < CG >=< CG > \bigoplus_{j,m,\mathbf{d}} D^{jm}_{\mathbf{h}'\mathbf{h}}(\Lambda, a)$$

$$\Downarrow$$

$$U_0(\Lambda, a)|(m, j), \mathbf{h}; \mathbf{d}\,\rangle = \sum \int |(m, j), \mathbf{h}'; \mathbf{d}\,\rangle D^{jm}_{\mathbf{h}'\mathbf{h}}(\Lambda, a)$$

Dynamics

$$\langle (m', j'), \mathbf{h}'; \mathbf{d}' \,|V|(m, j), \mathbf{h}; \mathbf{d}\,\rangle = \delta_{jj'}\delta_{\mathbf{h}\mathbf{h}'}\langle m'; \mathbf{d}' \| V^j \| m; \mathbf{d}\rangle$$

Find simultaneous eigenstates, $|(\bar{m}, j), \mathbf{h}; \bar{\mathbf{d}}\,\rangle$**, of**

$$\bar{M} = M_0 + V, j^2, \mathbf{h}$$

$\bar{M}, j^2, \mathbf{h}, \Delta\mathbf{h}$ **same commutation relations as** $M_0, j^2, \mathbf{h}, \Delta\mathbf{h}$

$$|(\bar{m}, j), \mathbf{h}; \bar{\mathbf{d}}\,\rangle \qquad \text{(complete)}$$

$$\boxed{\bar{U}(\Lambda, a)|(\bar{m}, j)\mathbf{h}; \bar{\mathbf{d}}\,\rangle = \sum \int d\mathbf{h}'|(\bar{m}, j), \mathbf{h}'; \bar{\mathbf{d}}\,\rangle D^{j\bar{m}}_{\mathbf{h}'\mathbf{h}}(\Lambda, a)}$$

Comparison with Galilean invariant quantum mechanics

$$H = \frac{P^2}{2M} + h \qquad h \qquad \leftrightarrow \qquad m$$

$$h = h_0 + V \qquad \leftrightarrow \qquad m = m_0 + V$$

$$\langle (h_0', j'), P, \mu; \mathbf{d}' \,|V|(h_0, j), P, \mu; \mathbf{d} \,\rangle =$$

$$\delta_{j'j} \underbrace{\delta_{\mu'\mu}\delta(P' - P)}_{\delta_{\mathbf{h'h}}} \langle \underbrace{\, h_0' \,}_{k'}; \mathbf{d}' \| V^j \| \underbrace{\, h_0 \,}_{k}; \mathbf{d} \rangle$$

Interactions

$$m = \sqrt{m_0^2 + 4 m v_{NN}} \qquad u = m - m_0$$

$$d\sigma = \frac{(2\pi)^4}{2k/m} |\langle k^+ | v_{NN} | k \rangle|^2 k^2 \frac{m}{2k} d\Omega =$$

$$\frac{(2\pi)^4}{2k/\omega} |\langle k^+ | u | k \rangle|^2 k^2 \frac{\omega}{2k} d\Omega$$

$$S(k_{nr}) = S(k_r)$$

Cluster properties

$$\Updownarrow$$

Poincaré invariance of isolated subsystems

Formulation of cluster properties

$$\mathbf{A}_i = \text{subsystem} \qquad \mathbf{S} = \text{system}$$

$$T_{\mathbf{A}_i}(x_i) := U_{\mathbf{A}_i}(I, x_i)$$

$$\mathbf{S} = \mathbf{A}_1 \cup \cdots \cup \mathbf{A}_m$$

$$\mathcal{H} = \mathcal{H}_{\mathbf{A}_1} \otimes \cdots \otimes \mathcal{H}_{\mathbf{A}_m}$$

$$\boxed{\left\| \left[U(\Lambda, a) - \bigotimes_{i=1}^{m} U_{\mathbf{A}_i}(\Lambda, a) \right] \bigotimes_{i=1}^{m} T_{\mathbf{A}_i}(x_i) |\psi\rangle \right\| \to 0}$$

as $(x_i - x_j)^2 \to +\infty$

Failure of cluster properties $(2+1$ system)

$$\mathbf{h} = (\vec{p}, j_{cz}) \qquad \vec{p}_3 = \vec{q} + \vec{P} F(m_{12}, \vec{q}, \vec{P})$$

$$\langle \vec{P}, \vec{q}, m_{12} | V | \vec{P}', \vec{q}', m'_{12} \rangle =$$
$$\delta(\vec{P} - \vec{P}')\delta(\vec{q} - \vec{q}')\langle m_{12} \| v(\vec{P}, \vec{q}) \| m_{12}' \rangle$$

$$\langle \vec{P}, \vec{q}, m_{12} | e^{i\vec{p}_3 \cdot \vec{a}} V e^{-i\vec{p}_3 \cdot \vec{a}} | \vec{P}', \vec{q}', m_{12}' \rangle =$$
$$\delta(\vec{P} - \vec{P}')\delta(\vec{q} - \vec{q}')\underbrace{e^{i\vec{P}\cdot\vec{a}[F(m_{12},\vec{q},\vec{P}) - F(m_{12}',\vec{q},\vec{P})]}\langle m_{12} \| v(\vec{P}, \vec{q}) \| m_{12}' \rangle}$$

$\rightarrow 0$ **as** $|\vec{a}| \rightarrow \infty$ **for** $\vec{P} \neq \vec{0}$.

Restoring cluster properties

$$U_{(12)(3)}(\Lambda, a) = \bar{U}_{(12)}(\Lambda, a) \otimes U_3(\Lambda, a)$$

(Satisfies cluster properties)

$$\Updownarrow$$

$$\bar{U}_{(12)(3)}(\Lambda, a)$$

($\bar{M}_{(12)(3)}$ Commutes with three-body $j^2, \mathbf{h}, \Delta\mathbf{h}$)

How are these representations related?

Interactions: $(2+1)$ system

$$V := M(m_{12}+v) - M(m_{12})$$

$$\langle \vec{P}, \vec{q}, \vec{k} | \bar{v} | \vec{P}\,', \vec{q}\,', \vec{k}\,' \rangle = \delta(\vec{P} - \vec{P}\,')\delta(\vec{q} - \vec{q}\,')\langle \vec{k} | v | \vec{k}\,' \rangle$$

$$\langle \vec{P}, \vec{p}_3, \vec{k} | v | \vec{P}\,', \vec{p}_3\,', \vec{k}\,' \rangle = \delta(\vec{P} - \vec{P}\,')\delta(\vec{p}_3 - \vec{p}_3\,')\langle \vec{k} | v | \vec{k}\,' \rangle$$

$$\left(m_{12} = \sqrt{m_1^2 + k^2} + \sqrt{m_2^2 + k^2} \qquad m_{12} \iff k^2\right)$$

$$\Downarrow$$

$$\langle \vec{P}, \vec{q}, \vec{k} | \bar{S} | \vec{P}\,', \vec{q}\,', \vec{k}\,' \rangle = \delta(\vec{P} - \vec{P}\,')\delta(\vec{q} - \vec{q}\,')\langle \vec{k} | s | \vec{k}\,' \rangle$$

$$\langle \vec{P}, \vec{p}_3, \vec{k} | S | \vec{P}\,', \vec{p}_3\,', \vec{k}\,' \rangle = \delta(\vec{P} - \vec{P}\,')\delta(\vec{p}_3 - \vec{p}_3\,')\langle \vec{k} | s | \vec{k}\,' \rangle$$

$$k^2 = k'^2$$

$$\Downarrow$$

$$\delta(\vec{P} - \vec{P}\,')\delta(\vec{p}_3 - \vec{p}_3\,') = \delta(\vec{P} - \vec{P}\,')\delta(\vec{q} - \vec{q}\,')$$

$$\Downarrow$$

$$\bar{S} = S$$

Ekstein's Theorem

$$WW^\dagger = I \qquad WMW^\dagger = \bar{M} \qquad M_0 = \bar{M}_0$$

$$\boxed{S = \bar{S} \Leftrightarrow \lim_{t \to \pm\infty} \|(W^\dagger - I)e^{-iM_0 t}|\psi\rangle\| = 0}$$

$$W = \bar{\Omega}_\pm(\bar{M}, M_0)\Omega_\pm^\dagger(M, M_0) + \bar{B}B^\dagger$$

(Ekstein 1960)

Dynamics related by W_p

$$S_p = \bar{S}_p \qquad (p := (ij)(k))$$

$$\Downarrow$$

$$W_p^\dagger \bar{M}_p W_p = M_p \qquad W_p = \bar{\Omega}_{p\pm} \Omega_{p\pm}^\dagger$$

$$\Downarrow$$

$$U_p(\Lambda, a)\Omega_{p+}\bar{\Omega}_{p+}^\dagger = \Omega_{p+} U_f(\Lambda, a)\bar{\Omega}_{+}^\dagger = \Omega_{p+}\bar{\Omega}_{p+}^\dagger \bar{U}_p(\Lambda, a)$$

$$\Downarrow$$

$$U_{ij}(\Lambda, a) \otimes U_k(\Lambda, a) = U_p(\Lambda, a) = W_p^\dagger \bar{U}_p(\Lambda, a) W_p$$

Summary: $2+1$ body problem

$$\begin{array}{ccc} U_{(12)_0}(\Lambda, a) \otimes U_3(\Lambda, a)\rangle & \xrightarrow{\langle AB|C\rangle_0} & U_{(12)_0(3)}(\Lambda, a) \\ V_{(12)(3)} \Big\downarrow & & \Big\downarrow \bar{V}_{(12)(3)} \\ U_{(12)_i}(\Lambda, a) \otimes U_3(\Lambda, a) & \xrightarrow[\langle AB|C\rangle_I]{} > & U_{(12)_i(3)}(\Lambda, a) \underbrace{\sim}_{W_p} \bar{U}_{(12)_i(3)}(\Lambda, a) \end{array}$$

There exist unitary transformations W that:

1. **Preserve S.**
2. **Do not preserve cluster properties.**
3. **Do not modify the description of a free asymptotic particle.**
4. **Form a group, and are elements of a norm closed algebra of asymptotic constants.**
5. $$\lim_{|\vec{a}_1-\vec{a}_2|\to\infty} \|(W_{12}-I)e^{i\vec{p}_1\cdot\vec{a}_1+i\vec{p}_2\cdot\vec{a}_2}|\psi\rangle = 0$$

These operators can be used to restore cluster properties in the interacting three-body system!

Construction of $U(\Lambda, a)$

$$\bar{U}_{p_1}(\Lambda, a) \otimes U_{p_2}(\Lambda, a) \underbrace{\longrightarrow}_{W_p} \bar{U}_p(\Lambda, a) \longrightarrow \left\{ \begin{array}{c} \bar{M}_p \\ j \\ \mathbf{h} \end{array} \right\}$$

$$\longrightarrow \left\{ \begin{array}{c} \bar{M} = \sum_p \bar{M}_p - 2M_0 \\ j \\ \mathbf{h} \end{array} \right\} \longrightarrow \bar{U}(\Lambda, a)$$

$$\underbrace{\longrightarrow}_{W^\dagger} U(\Lambda, a) = W^\dagger \bar{U}(\Lambda, a) W$$

$$\bar{U}(\Lambda, a) \to \bar{U}_p(\Lambda, a) = W_p(\otimes_i U_{p_i}(\Lambda, a)) W_p^\dagger \qquad W \to W_p$$

$$U(\Lambda, a) = W^\dagger \bar{U}(\Lambda, a) W \to W_p^\dagger W_p U_p(\Lambda, a) W_p^\dagger W_p = U_p(\Lambda, a)$$

Symmetric products

(Sokolov 1977)

$$W_p = \Omega_\pm(\bar{M}_p, M_0)\Omega_\pm^\dagger(M_p, M_0)$$

$$w_p := \ln(W_p)$$

$$w = \sum_p w_p \qquad p \in \{(12)(3), (23)(1), (31)(2)\}$$

$$W = e^w$$

$$\boxed{W \to W_p}$$

W is a scattering equivalence!

Practical three-body calculation

1. **Diagonalize (Coester 1965)**

$$\bar{M} := \bar{M}_1 + \bar{M}_2 + \bar{M}_3 - 2M_0, j^2, \mathbf{h}$$

$$\bar{M}|(j,\bar{m})\mathbf{h},\mathbf{d}\rangle = \bar{m}|(j,\bar{m})\mathbf{h},\mathbf{d}\rangle$$

2. **Transform eigenstates**

$$|(j,m)\mathbf{h},\mathbf{d}\rangle = W^{\dagger}|(j,\bar{m}),\mathbf{h},\mathbf{d}\rangle$$

3. **Transformation W is a scattering equivalence. Cannot change S matrix elements or bound state properties.**
4. **Observable consequences for $N \geq 4$.**

Practical three-body calculations

- Equations can be solved using standard Faddeev methods.
- The permutation operators in the relativistic case are Poincaré group Racah coefficients.
- Matrix elements of covariant Fields (like the em current) can be constructed using the Poincaré Wigner-Eckart theorem.

Currents

$$\bar{U}(\Lambda, a) = WU(\Lambda, a)W^{\dagger}$$

$$\bar{U}^{\dagger}(\Lambda, a)\bar{J}^{\mu}(0)\bar{U}(\Lambda, a) = \Lambda^{\mu}{}_{\nu}\bar{J}^{\nu}(0) \qquad [\bar{P}_{\mu}, \bar{J}^{\mu}(0)] = 0$$

$$U^{\dagger}(\Lambda, a)J^{\mu}(0)U(\Lambda, a) = \Lambda^{\mu}{}_{\nu}J^{\nu}(0) \qquad [P_{\mu}, J^{\mu}(0)] = 0$$

To maintain these relations in cluster limits U clusters $\Longleftrightarrow$ $J^{\mu}(x)$ clusters.

$$\Downarrow$$

$$\bar{J}^{\mu}(0) = WJ^{\mu}(0)W^{\dagger}$$

Structure of W_3

$$\langle p, k, p_3 | W_3 | p', k', p_3' \rangle =$$

$$\langle p, k, p_3 | [\bar{\Omega}\Omega^\dagger + \bar{B} B^\dagger] | p', k', p_3' \rangle =$$

$$\delta(p - p') \langle k | \delta(p_3 - p_3' - p(F(p, p_3, m(k)) - F(p, p_3, m_I)) | k' \rangle$$

$$(m_I = m(k) + v)$$

Summary

- Three-body bound state and scattering calculations based on Poincaré invariant quantum mechanics are possible (Elster, Glöckle, Golak, Kamada, Lin, Skibiński, Witała)
- Four-body systems are the simplest systems where corrections due to cluster properties can be seen in the scattering matrix.
- The construction of the exchange currents generated by restoring cluster properties was outlined.
- These importance of these corrections can now be tested. The required new input is W.