Poincaré invariant quantum scattering theory

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Outline

- Physics
- Introduction to quantum theory
- Symmetries and relativistic invariance
- Scattering theory
- Scattering equations (uniqueness and compactness)
- Treating functions of non-commuting operators
- Preliminary calculations

How do we see things that are smaller than an individual nucleon?

Measure scattered particles!

$$\Delta p \Delta x \geq \hbar \qquad \Delta p \geq \frac{\hbar}{\Delta x}$$

$\Delta x = 2 \times 10^{-16}$ **meters** $\Delta pc > 10^{9} ev \approx m_p c^2$

$$\Delta p \Delta x \approx \hbar$$

Requires quantum treatment

$$\Delta p c \approx m_p c^2$$

Requires relativistic treatment

Relativistic invariance is a symmetry

Introduction to Quantum Theory

Elements

- **Complex vector space (vectors represent states)**
- **Inner product**

$$(a, b)$$

- **Probabilities**

$$P_{a,b} := |(a, b)|^2$$

Symmetries are vector correspondences

$$a, b \rightarrow a', b'$$

that preserve

$$P_{ab} = |(a, b)|^2 = |(a', b')|^2 = P_{a'b'}$$

$$a' = Ua \qquad b' = Ub$$

Symmetry groups $g \in \mathcal{G}$:

$$U \rightarrow U(g)$$

$$U(g_1)U(g_2) = U(g_1 \cdot g_2)$$

The Poincaré group is the symmetry group of relativistic quantum theory

$$U \to U(g) \qquad g = (\Lambda^{\mu}{}_{\nu}, a^{\mu})$$

$$\Lambda \eta \Lambda^{t} = \eta \qquad \eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$g_1 \cdot g_2 = (\Lambda^{\mu}_{1\,\alpha} \Lambda^{\alpha}_{1\,\nu}, \Lambda^{\mu}_{2\,\alpha} a^{\alpha}_{1} + a^{\mu}_{2})$$

Time-translation symmetry

(subgroup of Poincaré group)

$$U(t)U(t') = U(t+t')$$

$$U(t) = e^{-iHt} \qquad a' = U(t)a$$

$$P_{ab} = |(a,b)|^2 = |(a',b')|^2 = P_{a'b'}$$

$$H = \sqrt{M^2 + \mathbf{P}^2} \qquad M = M_0 + V$$

Time-translation

$$H_0 = \sqrt{\mathbf{P}^2 + M_0^2} \leftrightarrow U_0(t) = e^{-iH_0 t}$$

$$H = \sqrt{\mathbf{P}^2 + M^2} \leftrightarrow U(t) = e^{-iHt}$$

$M =$ **Invariant Casimir operator of Poincaré group**

$\mathbf{P} =$ **Space translation generator (momentum)**

Formulation of the scattering problem

Initial and final conditions at $t = \pm\infty$

$$\lim_{t\to+\infty} \|[U(t)a - U_0(t)a_0]\| = 0$$

$$\lim_{t\to-\infty} \|[U(t)b - U_0(t)b_0]\| = 0$$

$$P_{ab} = |(a,b)|^2 = |(a_0, S b_0)|^2$$

$$S = \Omega_+^\dagger \Omega_- \qquad a = \Omega_+ a_0 \qquad b = \Omega_- b_0$$

$$\Omega_\pm = \lim_{t\to\pm\infty} U(-t)U_0(t)$$

How do we see the target?

Extract V from $|(a_0, S b_0)|^2$

Ekstein Theorem

$$H = H_0 + F(V) \qquad H' = AHA^{\dagger} = H_0 + F(V')$$

$$S = S(H, H_0) = S(H', H_0) = S'$$

$$\Updownarrow$$

$$AA^{\dagger} = I \qquad \lim_{t \to \pm\infty} \|(A - I)U_0(t)a_0\| = 0$$

V can only be measured up to "scattering equivalence"

Relation between V and S

$$\Omega_\pm = \lim_{t\to\pm\infty} e^{iHt} e^{-iH_0 t} = \lim_{t\to\pm\infty} e^{if(H)t} e^{-if(H)_0 t} = \lim_{t\to\pm\infty} e^{iMt} e^{-iM_0 t}$$

$$S = \Omega_+^\dagger \Omega_- \qquad M = M_0 + V$$

$$(a_0, Sb_0) = (a_0, b_0) - 2\pi i \delta(m_a - m_b)(a_0, T(m_a + i0^+) b_0)$$

$$M_0 a_0 = m_a a_0 \qquad M_0 b_0 = m_b b_0$$

Two-Particle Scattering (Operator Equations)

$$T(z) := V + V(z - M)^{-1} V$$

$$(z - M)^{-1} = (z - M_0)^{-1} + (z - M_0)^{-1} V (z - M)^{-1}$$

$$\boxed{T(z) = V + V(z - M_0)^{-1} T(z)}$$

$V(z - M_0)^{-1}$ **compact**

$$|||V(z - M_0)^{-1} - K_F||| < \epsilon$$

$$T(z) \approx (I - K_F)^{-1} V$$

$$T(z) \approx V + V(z - M_0)^{-1}(I - K_F)^{-1} V$$

Three-Particle Scattering (Operator Equations)

$$M = M_0 + W \qquad W = \sum_\alpha W_\alpha \qquad \alpha \in \{(12), (23), (31)\}$$

$$W_\alpha = M_\alpha - M_0$$

$$M_\alpha = \sqrt{(M_0(\alpha) + V_\alpha)^2 + \mathbf{q}_\alpha^2} + \sqrt{m^2 + \mathbf{q}_\alpha^2}$$

$$W^\alpha := \sum_{\beta \neq \alpha} W_\beta$$

$$(a_0, S^{\alpha\beta} b_0) = (a_0, b_0) - 2\pi i \delta(m_a - m_b)(a_0, T^{\alpha\beta}(m_a + i0^+) b_0)$$

$$T^{\alpha\beta}(z) = W^{\beta} + W^{\alpha}(z - M)^{-1} W^{\beta}$$

$$(z - M)^{-1} = (z - M_{\alpha})^{-1} + (z - M_{\alpha})^{-1} W^{\alpha} (z - M)^{-1}$$

$$\boxed{T^{\alpha\beta}(z) = W^{\beta} + W^{\alpha}(z - M_{\alpha})^{-1} T^{\alpha\beta}(z)}$$

Kernel not compact; solutions not unique ?

Faddeev

$$W^{\alpha} = \sum_{\beta \neq \alpha} W_{\beta}$$

$$W^{\alpha}(z - M)^{-1} = \sum_{\gamma \neq \alpha} W_{\gamma}(z - M_{\gamma})^{-1}[I + W^{\gamma}(z - M)^{-1}]$$

$$T^{\alpha\beta}(z) = W^{\beta} + W^{\alpha}(z - M)^{-1} W^{\beta}$$

$$\boxed{T^{\alpha\beta}(z) = W^{\beta} + \sum_{\gamma \neq \alpha} W_{\gamma}(z - M_{\gamma})^{-1} T^{\gamma\beta}(z)}$$

Iterated kernel of coupled equations compact

Elements of a calculation $(N = 2)$

$$\mathcal{H} = L^2(\mathbb{R}^3, d\mathbf{k}) \otimes L^2(\mathbb{R}^3, d\mathbf{P})$$

$$a \longrightarrow a(\mathbf{k}, \mathbf{P})$$

$$M_0\, a(\mathbf{k}, \mathbf{P}) = 2\sqrt{\mathbf{k}^2 + m^2}\, a(\mathbf{k}, \mathbf{P})$$

$$(V\, a)(\mathbf{k}, \mathbf{P}) = \int d\mathbf{k}'\, v(\mathbf{k}, \mathbf{k}')\, a(\mathbf{k}', \mathbf{P})$$

$$(T(z)\, a)(\mathbf{k}, \mathbf{P}) = \int d\mathbf{k}'\, t(\mathbf{k}, z, \mathbf{k}')\, a(\mathbf{k}', \mathbf{P})$$

$$z = z_0 = 2\sqrt{\mathbf{k}_0^2 + m^2} + i0^+$$

Interactions based on meson exchange

$$v(\mathbf{k}, \mathbf{k}') = \sum \frac{\lambda_i}{m_i^2 c^2 + (\mathbf{k} - \mathbf{k}')^2}$$

m_i = **mass of** i^{th} **meson.**

λ_i **coupling strength of** i^{th} **meson**

Elements of a calculation $(N = 2)$

$$t(\mathbf{k}', z_0, \mathbf{k}) = v(\mathbf{k}', \mathbf{k}) +$$

$$\int d\mathbf{k}'' \frac{v(\mathbf{k}', \mathbf{k}'')}{z_0 - 2\sqrt{\mathbf{k}''^2 + m^2}} t(\mathbf{k}'', z_0, \mathbf{k})$$

Integrable singularity at $|\mathbf{k}''| = |\mathbf{k}_0|$ for

$$z_0 := 2\sqrt{\mathbf{k}_0^2 + m^2} + i0^+$$

Elements of a calculation $(N = 3)$

$$\mathcal{H} = L^2(\mathbb{R}^3, d\mathbf{k}_\alpha) \otimes L^2(\mathbb{R}^3, d\mathbf{q}_\alpha) \otimes L^2(\mathbb{R}^3, d\mathbf{P})$$

$$a \longrightarrow a(\mathbf{k}_\alpha, \mathbf{q}_\alpha, \mathbf{P})$$

$$M_0 = \sqrt{4\mathbf{k}_\alpha^2 + 4m^2 + \mathbf{q}_\alpha^2} + \sqrt{m^2 + \mathbf{q}_\alpha^2}$$

$$W_\alpha = \sqrt{(2\sqrt{\mathbf{k}_\alpha^2 + m^2} + V_\alpha)^2 + \mathbf{q}_\alpha^2} - \sqrt{(2\sqrt{\mathbf{k}_\alpha^2 + m^2})^2 + \mathbf{q}_\alpha^2}$$

$$(V_\alpha\, a)(\mathbf{k}_\alpha, \mathbf{q}_\alpha, \mathbf{P}) = \int d\mathbf{k}'_\alpha v(\mathbf{k}_\alpha, \mathbf{k}'_\alpha)\, a(\mathbf{k}'_\alpha, \mathbf{q}_\alpha, \mathbf{P})$$

How do we compute the kernel of W_α?

Elements of a calculation $(N = 3)$

$$W_\gamma (z - M_\gamma)^{-1} = T_\gamma(z)(z - M_0)^{-1}$$

$$T_\gamma(z) = W_\gamma + W_\gamma (z - M_\gamma)^{-1} W_\gamma$$

$$T_\gamma(z) = W_\gamma + W_\gamma (z - M_0)^{-1} T_\gamma(z)$$

$$W_\gamma = \sqrt{(2\sqrt{\mathbf{k}_\gamma^2 + m^2} + V_\gamma)^2 + \mathbf{q}_\gamma^2} - \sqrt{(2\sqrt{\mathbf{k}_\gamma^2 + m^2})^2 + \mathbf{q}_\gamma^2}$$

Use the property that

$$r := 2\sqrt{\mathbf{k}_\gamma^2 + m^2} + V_\gamma$$

and

$$R := \sqrt{(2\sqrt{\mathbf{k}_\gamma^2 + m^2} + V_\gamma)^2 + \mathbf{q}_\gamma^2} = \sqrt{r^2 + \mathbf{q}_\gamma^2}$$

have the same eigenfunctions b_r with eigenvalues B_r and β_r

$$Rb_r = B_r b_r \qquad rb_r = \beta_r b_r$$

For $V_\gamma \to 0$ we have $R \to R_0$, $r \to r_0$, $B_r \to B_{r0}$ and $\beta_r \to \beta_{r0}$ with

$$R_0 b_{r0} = B_{r0} b_{r0} \qquad r_0 b_{r0} = \beta_{r0} b_{r0}$$

$$\Omega^\dagger W_\gamma = T_\gamma \qquad \Omega^\dagger v_\gamma = t_\gamma \qquad b_r = b_{r0}\Omega^\dagger$$

$$(b'_{r0}, T_\gamma(B'_{r0})b_{r0}) = (b'_r, W_\gamma b_{r0}) =$$

$$(b'_r, (R - R_0)b_{r0}) = (b'_r, \frac{B'^2_r - B^2_{r0}}{B'_r + B_{r0}} b_{r0}) =$$

$$(b'_r, \frac{R^2 - R_0^2}{B'_r + B_{r0}} b_{r0}) = (b'_r, \frac{r^2 - r_0^2}{B'_r + B_{r0}} b_{r0}) =$$

$$(b'_r, \frac{\beta'^2 - \beta_0^2}{B'_r + B_{r0}} b_{r0}) = \frac{\beta' + \beta_0}{B'_r + B_{r0}} (b'_r, \beta' - \beta_0 b_{r0}) =$$

$$\frac{\beta' + \beta_0}{B'_r + B_{r0}} (b'_r, V_\gamma b_{r0}) = \frac{\beta' + \beta_0}{B'_r + B_{r0}} (b'_{0r}, t_\gamma(\beta'_{0r}) b_{r0})$$

$$T_\gamma(z) \to T_\gamma(\mathbf{k}_\gamma, \mathbf{k}'_\gamma, \mathbf{q}_\gamma, z)$$

$$T_\gamma(\mathbf{k}_\gamma, \mathbf{k}'_\gamma, \mathbf{q}_\gamma, z_0) =$$

$$\frac{2(\sqrt{\mathbf{k}_\gamma^2 + m^2} + \sqrt{\mathbf{k}'^2_\gamma + m^2})}{\sqrt{4\mathbf{k}_\gamma^2 + 4m^2 + \mathbf{q}_\gamma^2} + \sqrt{4\mathbf{k}'^2_\gamma + 4m^2 + \mathbf{q}_\gamma^2}} t_\gamma(\mathbf{k}_\gamma, \tilde{z}_0, \mathbf{k}'_\gamma)$$

$$z_0 = \sqrt{4\mathbf{k}_\gamma^2 + 4m^2 + \mathbf{q}_\gamma^2} + \sqrt{m^2 + \mathbf{q}_\gamma^2} + i0^+$$

$$\tilde{z}_0 = 2\sqrt{\mathbf{k}_\gamma^2 + m^2} + i0^+$$

Need $T_\gamma(z)$ for general $z \neq z_0$

$$(z_1 - M_\gamma)^{-1} = (z_2 - M_\gamma)^{-1} + (z_1 - M_\gamma)^{-1}(z_2 - z_1)(z_2 - M_\gamma)^{-1}$$

$$T_\gamma(\mathbf{k}_\gamma, \mathbf{k}'_\gamma, \mathbf{q}_\gamma, z) = T_\gamma(\mathbf{k}_\gamma, \mathbf{k}'_\gamma, \mathbf{q}_\gamma, z_0) +$$

$$\int d\mathbf{k}''_\gamma T_\gamma(\mathbf{k}_\gamma, \mathbf{k}''_\gamma, \mathbf{q}_\gamma, z_0) \frac{z - z_0}{(z - M''_0)(z_0 - M''_0)} T_\gamma(\mathbf{k}''_\gamma, \mathbf{k}'_\gamma, \mathbf{q}_\gamma, z)$$

$$M''_0 = \sqrt{4\mathbf{k}''^2_\gamma + 4m^2 + \mathbf{q}^2_\gamma} + \sqrt{m^2 + \mathbf{q}^2_\gamma}$$

Computational Strategy

- **Solve integral equations for two body** $t(\mathbf{k}, \tilde{z}_0, \mathbf{k}')$
- **Use** $t(\mathbf{k}, \tilde{z}_0, \mathbf{k}')$ **to calculate** $T_\gamma(\mathbf{k}_\gamma, \mathbf{k}'_\gamma, \mathbf{q}_\gamma, z_0)$
- **Use** $T_\gamma(\mathbf{k}_\gamma, \mathbf{k}'_\gamma, \mathbf{q}_\gamma, z_0)$ **in first resolvent equation to calculate** $T_\gamma(\mathbf{k}_\gamma, \mathbf{k}'_\gamma, \mathbf{q}_\gamma, z)$
- **Use** $T_\gamma(\mathbf{k}_\gamma, \mathbf{k}'_\gamma, \mathbf{q}_\gamma, z)$ **to construct kernel** $T_\gamma(z - M_0)^{-1}$ **of Faddeev equation**

$$T^{\alpha\beta}(\mathbf{k}_\alpha, \mathbf{q}_\alpha, z, \mathbf{k}_\beta, \mathbf{q}_\beta) =$$

$$(1-\delta_{\alpha\beta})(z - \sqrt{4\mathbf{k}_\beta^2 + 4m^2 + \mathbf{q}_\beta^2} - \sqrt{m^2 + \mathbf{q}_\beta^2}) +$$

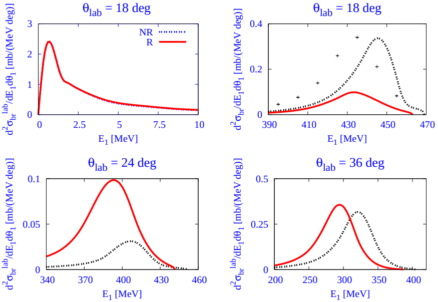
$$\sum_{\gamma\neq\alpha} \int d\mathbf{k}'_\gamma d\mathbf{k}''_\gamma d\mathbf{q}'_\gamma P(\mathbf{k}_\alpha, \mathbf{q}_\alpha, \mathbf{k}'_\gamma, \mathbf{q}'_\gamma) \times$$

$$\frac{T_\gamma(\mathbf{k}'_\gamma, \mathbf{k}''_\gamma, \mathbf{q}'_\gamma, z)}{z - \sqrt{4\mathbf{k}''^2_\gamma + 4m^2 + \mathbf{q}'^2_\gamma} - \sqrt{m^2 + \mathbf{q}'^2_\gamma}} T^{\gamma\beta}(\mathbf{k}''_\gamma, \mathbf{q}'_\gamma, z, \mathbf{k}_\beta, \mathbf{q}_\beta)$$

Numerical Issues

- **Matrices range from $10^6 \times 10^6$ to $10^7 \times 10^7$ (5 variables)**
- **Kernel is dense, has moving singularities.**
- **Kernel expressed in terms of solutions of subsystem singular integral equations.**
- **For larger problems the kernel cannot be stored.**
- **Solution methods - use iterations of kernel coupled with orthogonalization (Lanczos, Gram Schmidt) to reduce size of the linear system.**

$E_{lab} = 495$ MeV



Experiment (p,n) charge exchange cross section: X. Y. Chen, et. al. PRC 47, 2159 (1993)

Concluding Remarks

- Calculations of scattering cross sections for two and three-nucleon systems at energies sensitive to nucleon substructure are possible.
- Model interactions can be tested, but can only be determined up to an equivalence class.
- Compactness $\rightarrow$ the three-body kernel expressed in terms of solutions of subsystem singular integral equations.
- Initial calculation suggest experiments that may be sensitive to the difference between an relativistic and non-relativistic treatment.
- Preliminary calculations are completed treating the full three body kernel to first order.
- Full calculation is in progress.