Wigner's theorem

Wigner's theorem shows that correspondences between states that preserve all quantum probabilities are necessarily given by unitary or antiunitary operators. These correspondences also preserve expectation values and ensemble averages..

Theorem: Consider a correspondence between quantum states

$$|\chi\rangle \to |\chi'\rangle \tag{1}$$

that satisfies

$$|\langle\chi|\xi\rangle|^2 = |\langle\chi'|\xi'\rangle|^2 \tag{2}$$

for all states $|\chi\rangle$ and $|\xi\rangle$. Then it is possible to choose arbitrary phases in each ray so the correspondence is linear or antilinear.

Proof: The proof follows the one in K. Gottfried, p. 226-228, *Quantum Mechanics, Volume 1 Fundamentals*

Let $\{|\phi_n\rangle\}$ be an orthonormal basis and let $\{|\phi'_n\rangle\}$ denote the corresponding primed states.

The frist step is to show that $\{|\phi'_n\rangle\}$ is also an orthonormal basis. Since $\langle\phi_m|\phi_n\rangle = \delta_{mn}$ it follows from the conditions of the theorem that $|\langle\phi'_m|\phi'_n\rangle|^2 = |\langle\phi_m|\phi_n\rangle|^2 = \delta_{mn}$. Since $\langle\phi'_m|\phi'_m\rangle > 0$ it follows that $\langle\phi'_m|\phi'_n\rangle = \delta_{mn}$. This shows that the $|\phi'_n\rangle$ are orthonormal.

In addition to being orthonormal $\{|\phi'_n\rangle\}$ is also a basis. To show this by contradiction assumee that there is a vector $|c'\rangle \neq 0|$ that is orthogonal to all of the basis functions, $|\phi'_n\rangle$. Then it follows that $\langle\phi_n|c\rangle = \langle\phi'_n|c'\rangle = 0$, which implies $|c\rangle = 0$, which by the assumption of the theorem also requires $|c'\rangle = 0$, contradicting the assumption that $|c'\rangle \neq 0$. This show that $\{|\phi'_n\rangle\}$ is an orthonormal basis.

The next step in the proof of Wigner's theorem is to fix the arbitrary phases of the $\{|\phi'_n\rangle\}$. To do this for each $n \neq 1$ define

$$|\alpha_n\rangle = |\phi_1\rangle + |\phi_n\rangle. \tag{3}$$

It follows that

$$|\alpha'_n\rangle = \sum_m |\phi'_m\rangle\langle\phi'_m|\alpha'_n\rangle \tag{4}$$

Since $|\langle\phi'_m|\alpha'_n\rangle| = |\langle\phi_m|\alpha_n\rangle|$ there are only two non-zero terms in this expansion and both coefficients are phases

$$|\alpha'_n\rangle = e^{ia_n}|\phi'_1\rangle + e^{ib_n}|\phi'_n\rangle \tag{5}$$

which can be writen as

$$e^{-ia_n}|\alpha'_n\rangle = |\phi'_1\rangle + e^{i(b_n-a_n)}|\phi'_n\rangle \tag{6}$$

For $n \neq 1$ we absorb these phases in the transformed vectors

$$|\alpha''_n\rangle = e^{-ia_n}|\alpha'_n\rangle \tag{7}$$

$$|\phi''_n\rangle = e^{i(b_n-a_n)}|\phi'_n\rangle \tag{8}$$

With these phase choices it follows that

$$|\alpha''_n\rangle = |\phi'_1\rangle + |\phi''_n\rangle \tag{9}$$

Next let $|\chi\rangle$ be an arbitrary vector and use the completeness of the bases to write

$$|\chi\rangle = \sum_m c_m|\phi_m\rangle \tag{10}$$

$$|\chi'\rangle = \sum_m c'_m|\phi''_m\rangle \tag{11}$$

By assumption

$$|c_m|^2 = |\langle\phi_m|\chi\rangle|^2 = |\langle\phi''_m|\chi'\rangle|^2 = |c'_m|^2 \tag{12}$$

We also have

$$\langle\alpha_n|\chi\rangle = \langle\phi_1|\chi\rangle + \langle\phi_n|\chi\rangle = c_1 + c_n \tag{13}$$

and from

$$\langle\alpha''_n|\chi'\rangle = \langle\phi'_1|\chi'\rangle + \langle\phi''_n|\chi'\rangle = c'_1 + c''_n \tag{14}$$

The assumptions of the theorem imply

$$|c'_1 + c''_n|^2 = |\langle\alpha''_n|\chi'\rangle|^2 = |\langle\alpha_n|\chi\rangle|^2 = |c_1 + c_n|^2 \tag{15}$$

Multiplying everything out

$$|c'_1|^2 + |c'_n|^2 + c'_1c'^*_n + c'_nc'^*_1 = |c_1|^2 + |c_n|^2 + c_1c^*_n + c_nc^*_1 \tag{16}$$

The assumptions of the theorem imply that the first two terms on the right are identical to the first two terms on the left so they cancel. What remains is

$$0 = c'_1c'^*_n + c'_nc'^*_1 - c_1c^*_n - c_nc^*_1 \tag{17}$$

Multiply by c'_n to get

$$0 = c'_1|c'^*_n|^2 + (c'_n)^2c'^*_1 - c_1c^*_nc'_n - c_nc^*_1c'_n \tag{18}$$

which is a quadratic equation for c'_n:

$$0 = (c'_n)^2 - c'_n\frac{(c_1c^*_n + c_nc^*_1)}{c_1\prime *} + \frac{c'_1}{c_1\prime *}|c_n|^2 \tag{19}$$

The roots are

$$c'_n = c_n \qquad c'_n = \frac{c_1}{c^*_1}c^*_n \tag{20}$$

We are still free to redefine the phase of $|\chi\rangle$ so $c_1 = c^*_1$. With this change we get the following two possibilitiees

$$|\chi'\rangle = \sum_m c_m|\phi'_m\rangle \tag{21}$$

or

$$|\chi'\rangle = \sum_m c_m^* |\phi'_m\rangle \tag{22}$$

In the first case the correspondence is linear, while in the the second case it is antilinear. Since the norm is preserved in both cases, it is unitary or antiunitary.

It is also true that if the correespondence is unitary or antuintary for one vector, it must be unitary or antuintary for all vectors. To show this let

$$|\chi'\rangle = \sum c_n^* |\phi'_n\rangle \tag{23}$$

and

$$|\xi'\rangle = \sum d_n |\phi'_n\rangle. \tag{24}$$

Then

$$\langle\chi'|\xi'\rangle = \sum_m d_m c_m \tag{25}$$

$$\langle\chi|\xi\rangle = \sum_m d_m c_m^* \tag{26}$$

and

$$|\langle\chi'|\xi'\rangle|^2 = \sum_{mn} d_m c_m d_n^* c_n^* \neq \sum_{mn} |d_m|^2 |c_n|^2 = |\langle\chi|\xi\rangle|^2 \tag{27}$$

which contradicts the assumption of the theory. This completes the proof of Wigner's theorem.

To summarize what has been shown, note that if W represeents the correspondence

$$W[|\chi\rangle] = |\chi'\rangle \tag{28}$$

then if $\{|\phi_n\rangle\}$ is a basis we can choose phases so

$$uW[|\phi_n\rangle] = |\phi'_n\rangle' \tag{29}$$

and

$$W[\sum_m c_m |\phi_m\rangle] = \sum_m c_m |\phi'_m\rangle \tag{30}$$

or

$$W[\sum_m c_m |\phi_m\rangle] = \sum_m c_m^* |\phi'_m\rangle \tag{31}$$

Bargmann's Theorem

Wigner's theorem implies that a relativistically invariant quantum mechanics is defined by a unitary representation of the component of the Poincaré group connected to the identity. In general the group elements map rays to rays, so there is some freedom available to define phases. In general there is no reason the expect that the product of unitary representations of group elements has the same phase at the unitary representation of the products of group elements

$$U(\Lambda_2, a_2)U(\Lambda_1, a_1) = e^{i\phi} U(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2). \tag{32}$$

The phases of these transformations can be eliminated by redefining the unitary operators provided the Lie Algebra has no central charges and the group is simply connected. For the Poincaré group there are no central charges, and the group becomes simply connected if $SO(3:1)$ is replaced by $SL(2,\mathbb{C})$. The group $SL(2,\mathbb{C})$ is discussed in the next section.