29:5742 Homework 7
Due 3/21

1. Let $H = \frac{\mathbf{p}^2}{2m}$ and let
$$\langle \mathbf{r}|\psi\rangle = Ne^{-\alpha \mathbf{r}^2}$$
where $N = (\frac{2\alpha}{\pi})^{3/4}$ is a normalization constant. Compute
$$\langle \mathbf{r}|e^{-i\frac{Ht}{\hbar}}|\psi\rangle$$
This illustrates how wave packets spread out in time. This is responsible for the large time covnvergence in the Cook-Jauch conditions.

2. Show that the one dimensional Gaussian wave packet
$$\psi(x) = (\frac{2\alpha}{\pi})^{1/4}e^{-\alpha \mathbf{x}^2}$$
is a minimal uncertainty state, i.e. $\Delta x \Delta p = \frac{\hbar}{2}$. These are useful for making localized wave packets with minimal mean momenta.

3. Assume that $H_0 = \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2}$ and $H = H_0 + V(|\mathbf{r}_2 - \mathbf{r}_1|)$ where V is a short range interaction. Show the scattering operator $S = \Omega_+^\dagger \Omega_-$ commutes with the total linear and angular momentum operators. Also show that if $\mathbf{p}_i \to \mathbf{p}_i + m_i \mathbf{v}$ that the S operator does not change. These conditions imply that S is independent of the choice of inertial coordinate system.

4. Show that
$$\lim_{\epsilon\to 0} \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\epsilon}{(x-y)^2+\epsilon^2} f(y)dy = f(x)$$
This shows
$$\lim_{\epsilon\to 0} \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\epsilon}{(x-y)^2+\epsilon^2} = \delta(x-y)$$

5. What is $T\Omega_\pm T^{-1}$ where T is the time reversal operator?

6. Let z be a complex number with the property that $(z-H)^{-1}$ and $(z-H_0)^{-1}$ both exist, where $H = H_0 + V$. Prove the first and second resolvent identities:
$$(z_1 - H)^{-1} - (z_2 - H)^{-1} = (z_1 - H)^{-1}(z_2 - z_1)(z_2 - H)^{-1},$$
$$(z-H)^{-1} - (z-H_0)^{-1} = (z-H_0)^{-1}V(z-H)^{-1} = (z-H)^{-1}V(z-H_0)^{-1}.$$

Homework 7 solutions

① $\langle \bar{r} | e^{-iH_0 t/\hbar} | \psi \rangle =$

$$\int \langle \bar{r} | e^{-i\frac{p^2}{2m}\frac{t}{\hbar}} | \bar{r}' \rangle N e^{-\alpha r'^2} d^3r' =$$

$$\int \langle \bar{r} | \bar{p} \rangle \langle \bar{p} | \bar{r}' \rangle N e^{-i\frac{p^2}{2m}\frac{t}{\hbar} - \alpha r'^2} d^3p\, d^3r'$$

This is the product of 3 one dimensional integrals

$$= \frac{1}{2\pi\hbar} \int e^{iP_x(x-x')/\hbar - i\frac{P_x^2}{2m}\frac{t}{\hbar} - \alpha x'^2} dx'\, dP_x$$

$$= \frac{1}{2\pi\hbar} \underbrace{\int e^{-\alpha\left(x' + \frac{iP_x}{2\alpha\hbar}\right)^2}}_{\sqrt{\frac{\pi}{\alpha}}} {}^{- \frac{P_x^2}{4\alpha\hbar^2} - i\frac{P_x^2}{2m}\frac{t}{\hbar} + iP_x\frac{x}{\hbar}}$$

$$= \frac{1}{2\pi\hbar}\sqrt{\frac{\pi}{\alpha}} \int e^{-\underbrace{\left(\frac{1}{4\alpha\hbar^2} + \frac{it}{2m\hbar}\right)}_{\beta} P_x^2 + iP_x\frac{x}{\hbar}} dP_x$$

$$= \frac{1}{2\pi\hbar}\sqrt{\frac{\pi}{\alpha}} \underbrace{\int e^{-\beta\left(P_x - \frac{ix}{2\beta\hbar}\right)^2}}_{\sqrt{\frac{\pi}{\beta}}} {}^{- \frac{x^2}{4\beta\hbar^2}}$$

cubing this and including the normalization factor gives

$$\langle \vec{r} | e^{-i \frac{p^2}{2m} \frac{t}{\hbar}} | \psi \rangle =$$

$$N \left(\frac{1}{2\hbar\sqrt{\alpha\beta}} \right)^3 e^{-\frac{r^2}{4\beta\hbar^2}}$$

where

$$\beta = \frac{1}{4\alpha^2\hbar^2} + \frac{it}{2m\hbar}$$

The factor $(\beta)^{-3/2}$ shows that the denominator grows like $t^{-3/2}$ for large t

(2) $\langle x^2 \rangle = \left(\frac{2\alpha}{\pi}\right)^{1/2} \int e^{-2\alpha x^2} x^2 dx =$

$$\left(\frac{2\alpha}{\pi}\right)^{1/2} \left(-\frac{1}{2}\frac{d}{d\alpha}\right) \int e^{-2\alpha x^2} dx$$

$$\left(\frac{2\alpha}{\pi}\right)^{1/2} \left(-\frac{1}{2}\frac{d}{d\alpha}\right) \left(\frac{\pi}{2\alpha}\right)^{1/2} =$$

$$\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) \frac{\alpha^{1/2}}{\alpha^{3/2}} = \frac{1}{4\alpha}$$

$$\langle p^2\rangle = \left(\frac{2\alpha}{\pi}\right)^{1/2} \int e^{-\alpha x^2}\left(-\hbar^2 \frac{d^2}{dx^2}\right) e^{-\alpha x^2}$$

$$\left(\frac{2\alpha}{\pi}\right)^{1/2} \int e^{-\alpha x^2}\left(-\hbar^2 \frac{d}{dx}\left(-2\alpha x\, e^{-\alpha x^2}\right)\right)$$

$$\left(\frac{2\alpha}{\pi}\right)^{1/2} \int e^{-\alpha x^2}\left(2\alpha\hbar^2 - 4\alpha^2\hbar^2 x^2\right) e^{-\alpha x^2}$$

$$2\alpha\hbar^2 - 4\alpha^2\hbar^2 \times \frac{1}{4\alpha} = \alpha\hbar^2$$

$$\therefore \quad \Delta p^2 \, \Delta x^2 = \langle p^2\rangle\langle x^2\rangle = \frac{1}{4\alpha}\alpha\hbar^2 = \left(\frac{\hbar}{2}\right)^2$$

$$\Delta p \, \Delta x = \frac{\hbar}{2}$$

③ Since p_1^2 p_2^2 $|r_1 - r_2|^2$ are rotationally invariant

$\mathcal{U}(R) H U^{+}(R) = H$

$\mathcal{U}(R) H_0 \mathcal{U}^{+}(R) = \text{"}$

$\mathcal{U}(R)\, e^{iHt/\hbar}\, e^{-iH_0 t/\hbar}\, \mathcal{U}^{+}(R) =$

$e^{i\,\mathcal{U}(R) H \mathcal{U}^{+}(R) t/\hbar}\, e^{-i\,\mathcal{U}(R) H_0 t/\hbar\, \mathcal{U}^{+}(R)} =$

$e^{iHt/\hbar}\, e^{-iH_0 t/\hbar}$

This can be written as

$$\mathcal{U}(R)\, e^{iHt/\hbar}\, e^{-iH_0 t/\hbar} = e^{iHt/\hbar}\, e^{-iH_0 t/\hbar}\, \mathcal{U}(R)$$

since $\mathcal{U}(R) = e^{-i\bar{J}\cdot\bar{\theta}/\hbar}$ differentiate with respect to any component of $\bar{\theta}$ gives

$$[\bar{J},\, e^{iHt/\hbar} e^{-iH_0 t/\hbar}] = 0$$

$$[\bar{J},\, e^{-iH_0 t/\hbar} e^{2iHt/\hbar} e^{-iH_0 t/\hbar}] = 0$$

Letting $t \to \infty$ gives

$$[\bar{J}, S] = 0$$

The identical argument works replacing $\bar{J}$ by $\bar{P} = \bar{P}_1 + \bar{P}_2$

For $\bar{P}_i \to \bar{P}_i - m_i \bar{v}$

$$H_0 \to \frac{(P_1 + m_1\bar{v})^2}{2m_1} + \frac{(P_2 + m_2\bar{v})^2}{2m_2} =$$

$$\frac{P_1^2}{2m_1} + \frac{P_2^2}{2m_2} + \bar{P}\cdot\bar{v} + \frac{m_1 v^2}{2} + \frac{m_2 v^2}{2}$$

similarly

$$H \to H + \bar{p}\cdot\bar{v} + \tfrac{1}{2}m_1v^2 + \tfrac{1}{2}m_2v^2$$

$$\Omega \to \lim_{t\to\infty} e^{i(H + \bar{p}\cdot\bar{v} + \frac{1}{2}m_1v^2 + \frac{1}{2}m_2v^2)\frac{t}{\hbar}} \times e^{-i(H_0 + \bar{p}\cdot\bar{v} + \frac{1}{2}m_1v^2 + \frac{1}{2}m_2v^2)\frac{t}{\hbar}}$$

since $\bar{p}\cdot\bar{v}$ commutes with H, H_0 this becomes

$$= e^{iHt/\hbar} e^{-iH_0t/\hbar}$$

this shows that this transformation leaves $\Omega_\pm$ invariant and it follows that it leaves S invariant

4 $$\lim_{\epsilon\to 0} \int_{y-a}^{x+b} \frac{\epsilon}{(x-y)^2+\epsilon^2} f(y)\, dy$$

let $u = \left(\frac{y-x}{\epsilon}\right) \quad du = \frac{dy}{\epsilon}$

$u: \; -\frac{a}{\epsilon} \to \frac{b}{\epsilon}$

$$\lim_{\epsilon\to} \int_{-\frac{a}{\epsilon}}^{\frac{b}{\epsilon}} \frac{1}{\epsilon}\, \frac{1}{u^2+1} f(x+u\epsilon)\, \epsilon\, du =$$

$$\lim_{\epsilon\to 0} \int_{-\frac{a}{\epsilon}}^{\frac{b}{\epsilon}} \frac{du}{u^2+1} \times f(x+u\epsilon)$$

as long as $f(x+u\epsilon)$ is bounded by a function of x we can take the limit inside the integr

$$= f(x) \int_{-\infty}^{\infty} \frac{du}{u^2+1} = \pi f(x)$$

5 $$T e^{iHt/\hbar} e^{-iH_0 t/\hbar} T^{-1} =$$

$$e^{T(iHt/\hbar)T^{-1}} \; e^{T(-iH_0 t/\hbar)T^{-1}}$$

$$e^{-iTHT^{-1}t/\hbar} \; e^{iTH_0T^{-1}t/\hbar} =$$

$$e^{-iHt/\hbar} \; e^{iH_0 t/\hbar} \quad \Longrightarrow$$

$$T\Omega_{\pm}T^{-1} = \Omega_{\mp}$$

(6) multiply by

$$(Z_1-H)\left[(Z_1-H)^{-1}-(Z_2-H)^{-1}-(Z_1-H)^{-1}(Z_2-Z_1)\right.$$

$$\left.\times\,(Z_2-H)^{-1}\right](Z_2-H)$$

$$= Z_2-H - Z_1+H - Z_2-Z_1 = 0$$

similarly

$$(Z-H_0)\left[(Z-H_0)^{-1}-(Z-H_0)^{-1}-(Z-H_0)^{-1}V(Z-H)^{-1}\right](Z-H)$$

$$= Z-H_0-Z+H - V = 0$$