29:246 Second Mid Term Exam

4/10

1. Consider two particles of equal mass $m = 2\mu$ interacting via a delta-shell potential of range R and strength λ:

$$\langle \mathbf{r}|V|\mathbf{r}'\rangle = -\lambda\delta(\mathbf{r}-\mathbf{r}')\delta(r-R)$$

   a. Solve the Lippmann-Schwinger equation for the $l = 0$ partial wave

$$\langle r|k^-\rangle_l = \langle r|k\rangle_l + \int_0^\infty g_l(r,r',k)V(r')r'^2dr'\langle r'|k^-\rangle_l$$

   where

$$\langle r|k\rangle_l = \frac{4\pi i^l}{(2\pi\hbar)^{3/2}} j_l(\frac{kr}{\hbar})$$

$$g_l(r,r',k) = -\frac{2\mu k}{\hbar^3} j_l(\frac{kr_<}{\hbar})h_l^1(\frac{kr_>}{\hbar})$$

   b. What is the form of this wave function for $r > R$?

   c. What is the $l = 0$ scattering amplitude?

2. Consider the scattering of two particles with relative energy $E = \frac{\mathbf{k}^2}{2\mu}$, where the interaction is rotationally invariant and has range R.

   a. How many partial waves are needed to accurately calculate the cross section?

   Assume that you have calculated the partial wave transition matrix elements, $t_l := t_l(k, k, \frac{k^2}{2\mu} + i0^+)$.

   b. What are the phase shifts for each partial wave.

   c. What are the partial wave scattering amplitudes for each partial wave.

   d. Find the full scattering amplitude as a function of scattering angle.

3. Let V be a potential the form

$$\langle \mathbf{k}|V|\mathbf{k}'\rangle = -\lambda e^{-\alpha(k^2+(k')^2)}$$

   a. What is the Born approximation for the transition operator?

   b. What is the center of mass differential cross section in the Born approximation?

   c. What is the total cross section in the Born approximation.

4. Let $H = H_0 + V$. Assume H_0 and V are invariant under rotations, translations, and assume that under $\mathbf{x}' = \mathbf{x} - \mathbf{v}t$ where $\mathbf{v}$ is a constant velocity H_0 transforms like

$$H_0 \to H_0' + \mathbf{P}\cdot\mathbf{v} + \frac{\mathbf{v}^2}{2M} \qquad V \to V' = V$$

where $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$ and $M = m_1 + m_2$. Let

$$\Omega_\pm = \lim_{t\to\pm\infty} e^{iHt}e^{-iH_0t}$$

a. Show $\Omega_\pm$ commutes with rotations.

b. Show $\Omega_\pm$ commutes with translations.

c. Show $\Omega_\pm$ is invariant under constant velocity shifts

d. What can you say about how phase shifts transform under rotations, translations and constant velocity shifts?

$$\cos^2(\frac{\theta}{2}) = \frac{1+\cos(\theta)}{2} \qquad \sin^2(\frac{\theta}{2}) = \frac{1-\cos(\theta)}{2}$$

$$\sum_{m=-l}^{l} Y^l_m(\hat{\mathbf{k}}')Y^{l*}_m(\hat{\mathbf{k}}) = \frac{2l+1}{4\pi}P_l(\hat{\mathbf{k}}'\cdot\hat{\mathbf{k}}) \qquad P_l(1) = 1$$

(1a)

$$\langle r_0 | k_{0-} \rangle = \langle r_0 | k_0 \rangle + \int_0^\infty g_0(r r' k)\,(-\lambda\, \delta(r-R))\, R^2 \langle R_0 | k_0^- \rangle$$

$$\langle R_0 | k_{0-} \rangle = \langle R_0 | k_0 \rangle + g_0(R R k)(-\lambda) R^2 \langle R_0 | k_0^- \rangle$$

$$\langle R_0 k_{0-} \rangle = \frac{\langle R_0 | k_0 \rangle}{1 + \lambda g_0(R R k) R^2}$$

$$\langle r_0 | k_{0-} \rangle = \langle r_0 | k_0 \rangle - \frac{\lambda g_0(r R k) R^2 \langle R_0 | k_0 \rangle}{1 + \lambda g_0(R R k) R^2} =$$

$$\frac{4\pi i^\ell}{(2\pi\hbar)^{3/2}} \left\{ j_\ell\left(\frac{kr}{\hbar}\right) + \frac{\frac{2\mu k \lambda R^2}{\hbar^3} h_0^+\left(\frac{kr_>}{\hbar}\right) j_0\left(\frac{kr_<}{\hbar}\right) j_0\left(\frac{kR}{\hbar}\right)}{1 - \frac{2\mu k \lambda R^2}{\hbar^3} j_0\left(\frac{kR}{\hbar}\right) h_0^+\left(\frac{kR}{\hbar}\right)} \right\}$$

(1b) let $h_0^+\left(\frac{kr_>}{\hbar}\right) \to h_0^+\left(\frac{kr}{\hbar}\right)$ $\quad j_0\left(\frac{kr_<}{\hbar}\right) \to j_0\left(\frac{kR}{\hbar}\right)$

in the above expression

(1c) from part b $\quad h^+ \to \frac{e^{ikr/\hbar}}{kr/\hbar}$

$$f_0 = \frac{2\mu k \lambda R^2}{\hbar^3}\left(\frac{\hbar}{k}\right) \frac{j_0\left(\frac{kR}{\hbar}\right) j_0\left(\frac{kR}{\hbar}\right)}{1 - \frac{2\mu k \lambda R^2}{\hbar^3} j_0\left(\frac{kR}{\hbar}\right) h_0^+\left(\frac{kR}{\hbar}\right)}$$

(coefficient of $\frac{e^{ikr/\hbar}}{r}$)

2a) $\frac{kR}{\hbar} = N$

2b) $e^{i\delta_\ell} \sin \delta_\ell = -\pi k \mu t_\ell$

$$\boxed{e^{2i\delta_\ell} = 1 - 2\pi i k \mu t_\ell}$$

2c) $f_\ell = -(2\pi)^2 \mu \hbar t_\ell$

$= -(2\pi)^2 \mu \hbar \frac{1}{-\pi \mu k} e^{i\delta_\ell} \sin \delta_\ell$

$= \frac{4\pi\hbar}{k} e^{i\delta_\ell} \sin \delta_\ell$

2d) $F(\hat{k}', \hat{k}) = \sum Y_{\ell m}(\hat{k}') \frac{4\pi\hbar}{k} e^{i\delta_\ell} \sin \delta_\ell \, Y^*_{\ell m}(\hat{k})$

$= \sum \frac{2\ell+1}{4\pi} \cdot \frac{4\pi\hbar}{k} e^{i\delta_\ell} \sin \delta_\ell \, P_\ell(\hat{k}' \cdot \hat{k})$

$= \sum \frac{(2\ell+1)\hbar}{k} e^{i\delta_\ell} \sin \delta_\ell \, P_\ell(\hat{k}' \cdot \hat{k})$

3a) $\langle k' | T | k \rangle = \langle k' | V | k \rangle = -\lambda e^{-\alpha(k'^2 + k^2)}$

3b) $\frac{d\sigma}{d\Omega} = |F|^2 = (2\pi)^4 \hbar^2 \mu^2 \lambda^2 e^{-4\alpha k^2}$

3c) $\sigma = \int |F|^2 d\Omega(k) = (4\pi)(2\pi)^4 \hbar^2 \mu^2 \lambda^2 e^{-4\alpha k^2}$

a) $\mathcal{U}(R)\, e^{iHt/\hbar}\, e^{-iH_0t/\hbar}\, \mathcal{U}^\dagger(R) =$

$$e^{i\,\mathcal{U}(R)H\mathcal{U}^\dagger(R)t/\hbar}\, e^{-i\,\mathcal{U}(R)H_0\mathcal{U}^\dagger(R)t/\hbar} =$$

$$e^{iHt/\hbar}\, e^{-iH_0t/\hbar}$$

Taking limits $t \to \pm\infty$

$$\mathcal{U}(R)\,\Omega_\pm\,\mathcal{U}^\dagger(R) = \Omega_\pm$$

b) $T(\bar{a})\, e^{iHt/\hbar}\, e^{-iH_0t/\hbar}\, T^\dagger(\bar{a}) =$

$$e^{i\,T(a)HT^\dagger(a)t/\hbar}\, e^{-i\,T(a)H_0T^\dagger(a)t/\hbar} =$$

$$e^{iHt/\hbar}\, e^{-iH_0t/\hbar}$$

Taking limits

$$T(a)\,\Omega_\pm\, T^\dagger(a) = \Omega_\pm$$

c) $e^{i(H + \bar{p}\cdot\bar{v} + \frac{p^2}{2M})\frac{t}{\hbar}}\, e^{-i(H_0 + \bar{p}\cdot\bar{v} + \frac{p^2}{M2})\frac{t}{\hbar}} =$

$$e^{iHt/\hbar}\, e^{-iH_0t/\hbar}$$

since $\bar{p}$ commutes with H, H_0

this also preserves $\Omega_\pm$

d) since $e^{2i\delta} = S = \Omega_+\Omega_-$ this means that the phase shift is invariant under these transformations