Unitary one parameter groups

Unitary one parameter groups are sets of unitary operators, $U(\lambda)$, satisfying the following three conditions:

$$U(\lambda_1)U(\lambda_2) = U(\lambda_1 + \lambda_2) \tag{1}$$

$$U(0) = I \tag{2}$$

$$U^\dagger(\lambda_1) = U^{-1}(\lambda_1) = U(-\lambda_1) \tag{3}$$

Theorem: If $U(\lambda)$ is a unitary one parameter group then

$$\boxed{U(\lambda) = e^{-i\lambda G}} \tag{4}$$

where $G = G^\dagger$ and G is independent of λ.

Proof:

$$0 = \frac{d}{d\lambda} I = \frac{d}{d\lambda}(U(\lambda)U^\dagger(\lambda)) = U'(\lambda)U^\dagger(\lambda) + U(\lambda)U^{\dagger\prime}(\lambda) \tag{5}$$

where

$$U'(\lambda) := \frac{d}{d\lambda} U(\lambda). \tag{6}$$

It follows that

$$U'(\lambda)U^\dagger(\lambda) = -U(\lambda)U^{\dagger\prime}(\lambda) = -(U'(\lambda)U(\lambda))^\dagger. \tag{7}$$

Define

$$G(\lambda) = iU'(\lambda)U^\dagger(\lambda) \tag{8}$$

then (7) means

$$G(\lambda) = G^\dagger(\lambda) \tag{9}$$

To show that $G(\Lambda)$ is independent of λ let $\lambda = \lambda_1 + c$ where c is a constant. Then

$$\frac{d}{d\lambda} = \frac{d\lambda_1}{d\lambda}\frac{d}{d\lambda_1} = 1 \times \frac{d}{d\lambda_1} = \frac{d}{d\lambda_1} \tag{10}$$

$$G(\lambda) = iU'(\lambda))U^\dagger(\lambda) = i(\frac{d}{d\lambda_1}U(\lambda_1 + c))U^\dagger(\lambda_1 + c) =$$

$$i(\frac{d}{d\lambda_1}U(\lambda_1)U(c)U^\dagger(\lambda_1)U^\dagger(c) = i(\frac{d}{d\lambda_1}U(\lambda_1)U^\dagger(-c + \lambda_1 + c) =$$

$$i(\frac{d}{d\lambda_1}U(\lambda_1)U^\dagger\lambda_1) = G(\lambda_1) \tag{11}$$

which shows that G is independent of λ.

It follows that

$$U'(\lambda) = -iGU(\lambda) \qquad U(0) = I \tag{12}$$

which can be integrated to get

$$\boxed{U(\Lambda) := e^{-iG\lambda} := I + \sum_{n=1}^{\infty} \frac{(-iG\lambda)^n}{n!}} \tag{13}$$

which converges on eigenstates of G with finite eigenvalues.

Rotations about a fixed axis

In a quantum theory we expect that changes of coordinates by rotating about a fixed axis will not change quantum probabilities. A theorem due to Wigner (see Gottfied or Weinberg for an elementary proof) implies that transformations that preserve all probabilities must be either unitary or antiunitary.

The unitary or antiunitary, $\langle A\psi|A\phi\rangle = \langle\phi|\psi\rangle$, operator representing a rotations through an angle θ about a fixed axis (z axis for example) should form a unitary one parameter group, i.e.

$$U(\theta_2)U(\theta_1) = U(\theta_1 + \theta_2) \tag{14}$$

Exercise: Show that that if $U(\lambda_1 + \lambda_2) = U(\lambda_1)U(\lambda_2)$ then $U(\lambda)$ cannot be antiunitary.

Technical Comment: It is also possible that the products above could also include a phase that depends on $(\theta_1 + \theta_2)$. This is discussed in detail in Weinberg - in many cases the phase can be transformed away by redefining the operators. I will discuss this later; it only becomes non trivial when discussing non-relativistic boosts to coordinate systems moving with constant velocity.

Equation (14) means that $U(\theta) = e^{-i\theta G_z}$ where $G_z = G_z^\dagger$ is independent of θ. G_z is called the infinitesimal generator of rotations about the z axis.

In order to discuss rotations let $\mathbf{V}$ be a fixed vector in the $x - y$ plane. It can be expanded in a basis

$$\mathbf{V} = V_x\hat{\mathbf{x}} + V_y\hat{\mathbf{y}}. \tag{15}$$

If we use a new set of basis vectors related to the original basis by a rotation by an angle θ about the $\hat{\mathbf{z}}$ axis the new basis vectors can be expressed in terms of the original ones as

$$\hat{\mathbf{x}}' = \cos(\theta)\hat{\mathbf{x}} + \sin(\theta)\hat{\mathbf{y}} \tag{16}$$

$$\hat{\mathbf{y}}' = \cos(\theta)\hat{\mathbf{y}} - \sin(\theta)\hat{\mathbf{x}}. \tag{17}$$

The vector $\mathbf{V}$ can expressed in the new basis as

$$\mathbf{V} = V'_x\hat{\mathbf{x}}' + V'_y\hat{\mathbf{y}}'. \tag{18}$$

Exercise: Use (15) - (18) to show that the components of $\mathbf{V}$ in these two different coordinate systems are related by

$$\begin{pmatrix} V'_x \\ V'_y \\ V'_z \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}. \tag{19}$$

This is the passive point of view. Instead of keeping the vector fixed and rotating the basis, we could keep the basis fixed and physically rotate the vector. This is called the active view. This will be used in what follows. In this case the components of the rotated vector in the original basis is given by the inverse of the above rotation

$$\begin{pmatrix} V'_x \\ V'_y \\ V'_z \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}. \tag{20}$$

We still expect quantum probabilities to be conserved with respect to active rotations. Finally note that in order to preserve the group representation property for vector operators

$$U(R)\mathbf{V}U^\dagger(R) = R^t\mathbf{V} = \mathbf{V}R$$

so matrix multiplication is in the same order as multiplication by the unitary rotation operators:

$$\mathbf{V}' = U(R_2R_1)\mathbf{V}U(R_2R_1)^\dagger = U(R_2)U(R_1)\mathbf{V}U^\dagger(R_1)U^\dagger(R_2) =$$

$$R_1^tU(R_2)\mathbf{V}U^\dagger(R_2) = R_1^tR_2^t\mathbf{V} = (R_2R_1)^t\mathbf{V} = VR_2R_1. \tag{21}$$

If $\mathbf{V}'$ represents components of a rotated vector operator on the Hilbert space (like a coordinate or momentum operator) then in terms of the generator G_z

$$\mathbf{V}' = U(\theta)\mathbf{V}U^\dagger(\theta) =$$

$$e^{-iG_z\theta}\mathbf{V}e^{iG_z\theta} = R^t\mathbf{V} = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \tag{22}$$

Exercise: Differentiate equation (22) with respect to θ and then set θ to zero to show:

$$-[iG_z, \mathbf{V}] = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}. \tag{23}$$

Component by component this gives

$$[G_z, V_x] = iV_y, \qquad [G_z, V_x] = -iV_x, \qquad [G_z, V_z] = 0 \tag{24}$$

where G_z is the infinitesimal generator of rotations about the z axis.

This can be repeated for rotations about the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ axes. The results are for rotations about the $\hat{\mathbf{x}}$ axis

$$[G_x, V_x] = 0 \qquad [G_x, V_y] = iV_z, \qquad [G_x, V_z] = -iV_y, \tag{25}$$

and for rotations about $\hat{\mathbf{y}}$ axis

$$[G_y, V_x] = -iV_z, \qquad [G_y, V_y] = 0 \qquad [G_y, V_x] = iV_x, \tag{26}$$

These relations can be summarized by

$$\boxed{[G_i, V_j] = i\sum_{k=1}^{3} \epsilon_{ijk} V_k} \tag{27}$$

where (i, j, k) represents (x, y, z), (y, z, x) or (z, x, y) and ϵ_{ijk} is completely antisymmetric and $\epsilon_{123} = \epsilon_{xyz} 1$.

The commutation relations characterize any operator whose components transform like a vector when the coordinate system is rotated.

SU(2)

Vectors can be represented by 2×2 traceless Hermitian matrices

$$\mathbf{V} = (V_x, V_y, V_z) = (V_1, V_2, V_3) \tag{28}$$

$$\boxed{V = \mathbf{V} \cdot \boldsymbol{\sigma} = V_x \sigma_x + V_y \sigma_y + V_z \sigma_z = \begin{pmatrix} V_3, V_1 - iV_2 \\ V_1 + iV_2, -V_3 \end{pmatrix} \qquad V_i = \frac{1}{2}\mathrm{Tr}(\sigma_i V)} \tag{29}$$

Since

$$\det(V) = -V_x^2 - V_y^2 - V_z^2 = -\mathbf{V}^2 \tag{30}$$

transformations of the form

$$V' = WVW^\dagger \tag{31}$$

where W is a unitary 2×2 matrix with determinant 1 preserve the Hermiticity, trace and determinant. These are called $SU(2)$ matrices. For $V_i' = \sum_{j=1}^{3} R_{ij} V_j$ where R_{ij} is a rotation matrix

$$V' = \sum_{i,j=1}^{3} \sigma_i R_{ij} V_j = (W \sum_{j=1}^{3} V_j \sigma_j W^\dagger). \tag{32}$$

Multiply both sides by σ_l and take traces to get

$$V_l' = R_{lj} V_j = \sum_j \frac{1}{2}\mathrm{Tr}(\sigma_l W \sigma_j W^\dagger) V_j \tag{33}$$

Differentiating (33) with respect to V_j gives an expression for the $SO(3)$ matrix R in terms of the $SU(2)$ matrices W:

$$\boxed{R_{lj} = \frac{1}{2}\mathrm{Tr}(\sigma_l W \sigma_j W^\dagger)} \tag{34}$$

Replacing W by $-W$ does not change the result. This gives a 2 to 1 correspondence between $SU(2)$ matrices $\pm W$ and real orthogonal matrices R_{ij} with determinant 1 (rotations).

Exercise: Show that a general $SU(2)$ matrix can be expressed in the form

$$\boxed{W = e^{-i\frac{1}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}} = \cos(\frac{\theta}{2})I - i\hat{\boldsymbol{\theta}}\sin(\frac{\theta}{2})}. \tag{35}$$

Exercise: Show that a general $SU(2)$ matrix can be expressed in the form

$$W = e_0 I + i\mathbf{e}\cdot\boldsymbol{\sigma} \qquad e_0^2 + \mathbf{e}\cdot\mathbf{e} = 1. \tag{36}$$

Exercise: Use (33) to show that for

$$W = e^{-i\frac{1}{2}\theta\sigma_x} \tag{37}$$

that

$$R_{lj} = \frac{1}{2}\mathrm{Tr}(\sigma_l W \sigma_j W^\dagger) \tag{38}$$

gives R corresponding to a rotation (20) about the $\hat{\mathbf{z}}$ axis.

The $SU(2)$ matrices represent spin $\frac{1}{2}$ rotations.

Angular momentum

Exercise: Show that if a Hamiltonian is rotationally invariant, $U(\theta)HU^\dagger(\theta) = H$ then $[G, H] = 0$ and G is a conserved quantity.

This means that for a rotationally invariant Hamiltonian $\mathbf{G} := (G_x, G_y, G_z)$ is conserved. This conserved quantity is identified with the angular momentum. Classically we expect the angular momentum operator to be a vector operator. Replacing V_i by G_i in (27) gives

$$[G_i, G_k] = i\sum_{k=1}^{3}\epsilon_{ijk}G_k \tag{39}$$

Note that since $U(\theta) = e^{-iG\theta}$ the quantity $G\theta$ must be dimensionless. Since angles are dimensionless. Since angular momentum has units (coordinate)(momentum) the angular momentum operator $J = \hbar G$ where $\hbar$ has units of angular momentum. Then

$$[J_i, J_k] = i\hbar\sum_{k=1}^{3}\epsilon_{ijk}J_k \tag{40}$$

In what follows I will choose units where $\hbar = 1$. In these units $\mathbf{J} = \mathbf{G}$ and

$$\boxed{[J_i, J_k] = i\sum_{k=1}^{3}\epsilon_{ijk}J_k} \tag{41}$$

We expect that the length of $\mathbf{J}$ is invariant with respect to rotations. The length squared or the angular momentum is defined by

$$\mathbf{J}^2 := \mathbf{J}\cdot\mathbf{J} = J_xJ_x + J_yJ_y + J_zJ_z. \tag{42}$$

Exercise: Show that (35) implies

$$[J_i, \mathbf{J}^2] = 0 \tag{43}$$

which gives the expected property that the length of $\mathbf{J}$ remains invariant with respect to rotations.
Exercise: Show that $\mathbf{J}^2 = (\mathbf{J}^2)^\dagger$ follows from (41) and (42).

Note that while the individual components of $\mathbf{J}$ do not commute, any component of $\mathbf{J}$ commutes with $\mathbf{J}^2$. Since these are commuting Hermitian operators it is possible to find simultaneous eigenstates, $|\lambda, \mu\rangle$, of each one:

$$J_z|\eta, \mu\rangle = \mu|\eta, \mu\rangle \tag{44}$$

$$\mathbf{J}^2|\eta, \mu\rangle = \eta|\eta, \mu\rangle \tag{45}$$

It is useful to define

$$J_\pm = J_x \pm iJ_y \tag{46}$$

Exercise: Show that the commutation relations (41) imply

$$\boxed{[J_z, J_\pm] = \pm J_\pm} \tag{47}$$

It follows from (47) that

$$J_z J_\pm|\eta, \mu\rangle = J_\pm J_z|\eta, \mu\rangle \pm J_\pm|\eta, \mu\rangle = (\mu \pm 1)J_\pm|\eta, \mu\rangle \tag{48}$$

This means that if $|\eta, \mu\rangle$ is an eigenstate of J_z with eigenvalue μ then $J_\pm|\eta, \mu\rangle$ is zero or and eigenstate of J_z with eigenvalue $\mu \pm 1$. Because of this $J_\pm$ are called raising and lowering operators.
Exercise: Show that

$$J_\mp J_\pm = \mathbf{J}^2 - J_z J_z \mp J_z \tag{49}$$

which gives

$$\mathbf{J}^2 = \frac{1}{2}(J_+ J_- + J_- J_+) + J_z J_z \tag{50}$$

Assuming that the states $|\eta, \mu\rangle$ are normalized to unity

$$1 = \||\eta, \mu\rangle\|^2 = \langle\eta, \mu|\eta, \mu\rangle \tag{51}$$

then

$$\|J_\pm|\eta, \mu\rangle\|^2 = \langle\eta, \mu|J_\mp J_\pm \eta, \mu\rangle = \langle\eta, \mu|\mathbf{J}^2 - J_z^2 \mp J_z \eta, \mu\rangle =$$
$$(\eta - \mu^2 \mp \mu)\langle\eta, \mu|\eta, \mu\rangle(\eta - \mu^2 \mp \mu'). \tag{52}$$

Since this is the norm of a Hilbert space vector it must be non-negative

$$(\eta - \mu^2 \mp \mu) = (\eta - \mu(\mu \pm 1) \geq 0. \tag{53}$$

From this expression it is clear that as $\pm\mu$ increases for fixed η it eventually becomes negative. This means that there is a highest $\mu = \mu_{max}$ and lowest $\mu = \mu_{min}$ that makes this vanish:

$$J_+|\eta, \mu_{max}\rangle = J_-|\eta, \mu_{min}\rangle = 0. \tag{54}$$

This gives

$$\eta = \mu_{max}(\mu_{max} + 1) = \mu_{min}(\mu_{min} - 1). \tag{55}$$

Solving the quadratic equation for μ_{max} in terms of μ_{min} gives

$$\mu_{max}^2 + \mu_{max} - \mu_{min}(\mu_{min} - 1) = 0 \tag{56}$$

$$\mu_{max} = \frac{1}{2}(-1 \pm \sqrt{1 + 4\mu_{min}^2 - 4\mu_{min}}) = \frac{1}{2}(-1 \pm \sqrt{(1 - 2\mu_{min})^2}) =$$
$$\begin{cases} -\mu_{min} \\ \mu_{min} - 1 \end{cases} \tag{57}$$

The lower root has $\mu_{min} = \mu_{max} + 1$ which violates $\mu_{max} \geq \mu_{min}$. Therefore we must have

$$\boxed{\mu_{max} = -\mu_{min}} \tag{58}$$

and

$$\boxed{\eta = \mu_{max}(\mu_{max} + 1).} \tag{59}$$

Since $\mu_{max} - \mu_{min} = 2\mu_{max}$ must be an integer, it follows that the possible values of μ_{max} are $\mu_{max} = \frac{n}{2}$.

It conventional to define $j := \mu_{max}$ so $\eta = j(j+1)$. In this notation

$$|j, \mu\rangle := |\eta, \mu\rangle \tag{60}$$

$$\mathbf{J}^2 |j, \mu\rangle = j(j+1)|j, \mu\rangle \tag{61}$$

where $\eta = j(j+1)$ and $-j \leq \mu \leq j$.

Since $J_\pm$ increases or decreases μ it follows that

$$J_\pm |j, \mu\rangle = |j, \mu \pm 1\rangle N_\pm \tag{62}$$

where $N_\pm$ is a normalization constant. To compute $N_\pm$ use (49) to get

$$|N_\pm|^2 = \langle j, \mu | J_{-\mp} J_\pm | j, \mu\rangle =$$
$$\langle j, \mu | \mathbf{J}^2 - J_z^2 \mp J_z | j, \mu\rangle = j(j+1) - \mu(\mu \pm 1) = (j \mp \mu)(j \pm \mu + 1). \tag{63}$$

This determines the normalization constant up to a phase. Choosing the normalization constant to be real and positive gives

$$N_\pm = \sqrt{j(j+1) - \mu(\mu \pm 1)} = \sqrt{(j \mp \mu)(j \pm \mu + 1)} \tag{64}$$

and the relations

$$\boxed{J_\pm |j, \mu\rangle = |j, \mu \pm 1\rangle \sqrt{j(j+1) - \mu(\mu \pm 1)} = |j, \mu \pm 1\rangle \sqrt{(j \mp \mu)(j \pm \mu + 1)}}. \tag{65}$$

It follows that all states for fixed j and be calculated given $|j, j\rangle$ or $|j, -j\rangle$. The state $|j, j\rangle$ is called the highest weight state.

Schwinger method - finite dimensional representations of the rotation group

Since

$$[\mathbf{J}^2, \mathbf{J}] = 0, \tag{66}$$

$$\mathbf{J}^2 e^{-i\hat{\mathbf{n}}\cdot\mathbf{J}\theta}|j,\mu\rangle = e^{-i\hat{\mathbf{n}}\cdot\mathbf{J}\theta}\mathbf{J}^2|j,\mu\rangle = j(j+1)e^{-i\hat{\mathbf{n}}\cdot\mathbf{J}\theta}|j,\mu\rangle \tag{67}$$

which means that $e^{-i\hat{\mathbf{n}}\cdot\mathbf{J}\theta}|j,\mu\rangle$ is also an eigenstate of $\mathbf{J}^2$ with eigenvalue $j(j+1)$. It follows that

$$e^{-i\boldsymbol{\theta}\cdot\mathbf{J}}|j,\mu\rangle = \sum_{\nu=-j}^{j} |j,\nu\rangle\langle j,\nu|e^{-i\boldsymbol{\theta}\cdot\mathbf{J}}|j,\mu\rangle := \sum_{\nu=-j}^{j} |j,\nu\rangle D^j_{\nu\mu}(R(\boldsymbol{\theta})) \tag{68}$$

where

$$D^j_{\nu\mu}(R(\boldsymbol{\theta})) := \langle j,\nu|e^{-i\boldsymbol{\theta}\cdot\mathbf{J}}|j,\mu\rangle \tag{69}$$

is a $2j+1$ dimensional matrix representation of the rotation group. Here $R(\boldsymbol{\theta})$ is a $SU(2)$ matrix representing a rotation by an angle θ about the $\hat{\boldsymbol{\theta}}$ axis. It is normally referred to as a Wigner D-matrix. This means

$$\sum_{\alpha=-j}^{j} D^j_{\mu\alpha}(R_1)D^j_{\alpha\nu}(R_2) = D^j_{\mu\nu}(R_1R_2) \tag{70}$$

$$D^j_{\mu\nu}(R^{-1}) = (D^j_{\nu\mu}(R))^* \tag{71}$$

$$D^j_{\mu\nu}(I) = \delta_{\mu\nu}. \tag{72}$$

In what follows a method of Schwinger is used to compute $D(R)$. Start by defining

$$n_\pm := j \pm \mu \tag{73}$$

$$j = \frac{1}{2}(n_+ + n_-) \qquad \mu = \frac{1}{2}(n_+ - n_-). \tag{74}$$

Since there is a 1-1 correspondence between n_+, n_- and j,μ for this application it is useful to label the eigenstates of $\mathbf{J}^2$ and J_z using n_+, n_- instead of j,μ:

$$|n_+, n_-\rangle := |j,\mu\rangle. \tag{75}$$

It follows from (65) and (73) that

$$J_+|n_+,n_-\rangle = \sqrt{(n_+ + 1)n_-}|n_+ + 1, n_- - 1\rangle \tag{76}$$

$$J_-|n_+,n_-\rangle = \sqrt{n_+(n_- + 1)}|n_+ - 1, n_- + 1\rangle. \tag{77}$$

Define non-Hermitian operators $a_\pm$ by

$$a_\pm|0,0\rangle = 0 \tag{78}$$

$$a_+|n_+,n_-\rangle = \sqrt{n_+}|n_+ - 1, n_-\rangle \tag{79}$$

$$a_+^\dagger|n_+,n_-\rangle = \sqrt{n_+ + 1}|n_+ + 1, n_-\rangle \tag{80}$$

$$a_-|n_+,n_-\rangle = \sqrt{n_-}|n_+, n_- - 1\rangle \tag{81}$$

$$a_-^\dagger|n_+,n_-\rangle = \sqrt{n_- + 1}|n_+, n_- + 1\rangle. \tag{82}$$

Exercise: Show

$$[a_+, a_+^\dagger] = [a_-, a_-^\dagger] = 1 \tag{83}$$

$$[a_+, a_-^\dagger] = [a_-, a_+^\dagger] = [a_+, a_-] = [a_-^\dagger, a_+^\dagger] = 0 \tag{84}$$

Exercise: Show

$$a_\pm^\dagger a_\pm|n_+,n_-\rangle = n_\pm|n_+,n_-\rangle \tag{85}$$

These operators can be used to represent J_z and $J_\pm$ using (74,76-77):

$$J_\pm = a_\pm^\dagger a_\mp \tag{86}$$

$$J_z = \frac{1}{2}(a_+^\dagger a_+ - a_-^\dagger a_-) = \frac{1}{2}(a_+^\dagger, a_-^\dagger)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_+ \\ a_- \end{pmatrix} \tag{87}$$

It follows from (46) that

$$J_x = \frac{1}{2}(J_+ + J_-) \tag{88}$$

$$J_y = \frac{i}{2}(J_- - J_+) \tag{89}$$

Exercise: Show

$$J_x = \frac{1}{2}(a_+^\dagger a_- + a_-^\dagger a_+) = \frac{1}{2}(a_+^\dagger, a_-^\dagger)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_+ \\ a_- \end{pmatrix} \tag{90}$$

$$J_y = \frac{i}{2}(a_-^\dagger a_+ - a_+^\dagger a_-) = \frac{1}{2}(a_+^\dagger, a_-^\dagger)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_+ \\ a_- \end{pmatrix} \tag{91}$$

It is useful to use a matrix representation

$$a := \begin{pmatrix} a_+ \\ a_- \end{pmatrix} \qquad a^\dagger \begin{pmatrix} a_+, a_- \end{pmatrix} \tag{92}$$

With this notation equations (87)-(91)can be expressed as

$$\mathbf{J} = \frac{1}{2}a^\dagger \boldsymbol{\sigma} a \tag{93}$$

and

$$U(R(\boldsymbol{\theta})) = e^{-i\boldsymbol{\theta}\cdot\mathbf{J}} = e^{-\frac{i}{2}a^\dagger\hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma}a\theta} \tag{94}$$

This can be used to calculate

$$D^i_{\mu\nu}(R(\boldsymbol{\theta}) = \langle n'_+ n'_-|e^{-\frac{i}{2}a^\dagger\hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma}a\theta}|n_+ n_-\rangle =$$

$$\langle 0,0|\frac{a_+^{n'_+}}{\sqrt{n'_+!}}\frac{a_-^{n'_-}}{\sqrt{n'_-!}}e^{-\frac{i}{2}a^\dagger\hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma}a\theta}\frac{(a_+^\dagger)^{n_+}}{\sqrt{n_+!}}\frac{(a_-^\dagger)^{n_-}}{\sqrt{n_-!}}|0,0\rangle =$$

$$\langle 0,0| \frac{a_+^{n'_+}}{\sqrt{n'_+!}} \frac{a_-^{n'_-}}{\sqrt{n'_-!}} \frac{\left(e^{-\frac{i}{2}a^\dagger \hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma} a\theta} a_+^\dagger e^{\frac{i}{2}a^\dagger \hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma} a\theta}\right)^{n_+}}{\sqrt{n_+!}} \times$$

$$\frac{\left(e^{-\frac{i}{2}a^\dagger \hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma} a\theta} a_-^\dagger e^{\frac{i}{2}a^\dagger \hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma} a\theta}\right)^{n_-}}{\sqrt{n_-!}} \times \underbrace{e^{-\frac{i}{2}a^\dagger \hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma} a\theta}|0,0\rangle}_{|0,0\rangle}. \tag{95}$$

In order to compute

$$e^{-\frac{i}{2}a^\dagger \hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma} a\theta} a_\pm^\dagger e^{\frac{i}{2}a^\dagger \hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma} a\theta} \tag{96}$$

note that for general non commuting operators A and B if we define

$$X(\lambda) := e^{\lambda A} B e^{-\lambda A} \tag{97}$$

it follows that

$$\frac{d}{d\lambda} X(\lambda) = [A, X(\lambda)] \tag{98}$$

and by mathematical induction

$$\frac{d^n}{d\lambda^n} X(\lambda) = \underbrace{[A[\cdots[A, X(\lambda)]\cdots]}_{n \text{ times}}. \tag{99}$$

Using the definition of the exponential (13) of an operator gives

$$X(\lambda) = B + \sum_{n=1}^{\infty} \underbrace{[A[\cdots[A, B]\cdots]}_{n \text{ times}} \frac{\lambda^n}{n!} \tag{100}$$

Setting $\lambda = 1$ gives

$$e^A B e^{-A} = B + \frac{1}{1!}[A,B]\frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]]\cdots. \tag{101}$$

We use this to evaluate

$$e^{-\frac{i}{2}a^\dagger \hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma} a\theta} a_\pm^\dagger e^{\frac{i}{2}a^\dagger \hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma} a\theta} =$$

$$a_\pm^\dagger - \sum_{jl} \frac{i\theta}{2}[a_j^\dagger (\hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma})_{jk} a_k, a_\pm^\dagger] +$$

$$\frac{\theta^2}{2^2 2!}\sum([a_l^\dagger(\hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma})_{lm} a_m, [a_k^\dagger(\hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma})_{kj} a_j, a_\pm^\dagger]] + \cdots =$$

$$a_\pm^\dagger - \sum_i \frac{i\theta}{2} a_i^\dagger (\hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma})_{i\pm} + \frac{1}{2!}\sum_i (-\frac{i\theta}{2})^2 a_i^\dagger (\hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma})^2_{i\pm} + \frac{1}{3!}\sum_i (-\frac{i\theta}{2})^3 a_i^\dagger (\hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma})^3_{i\pm} + \cdots \tag{102}$$

Since

$$(\hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma})^2 = \sum_{ij} \hat{\boldsymbol{\theta}}_i \hat{\boldsymbol{\theta}}_j \sigma_i \sigma_j = \sum_{ij} \hat{\boldsymbol{\theta}}_i \hat{\boldsymbol{\theta}}_j (\delta_{ij} + i\sum_l \epsilon_{ijk}\sigma_k) = \sum_i \hat{\boldsymbol{\theta}}_i \hat{\boldsymbol{\theta}}_i = 1 \tag{103}$$

(102) can be summed explicitly

$$e^{-\frac{i}{2}a^\dagger \hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma} a\theta} a^\dagger_\pm e^{\frac{i}{2}a^\dagger \hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma} a\theta} =$$

$$\cos(\frac{\theta}{2})a^\dagger_\pm - i\sum_{j=\pm} \sin(\frac{\theta}{2})a^\dagger_j(\hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma})_{j\pm} =:$$

$$\sum_{k=\pm} a^\dagger_k R_{k\pm} = a^\dagger_+ R_{+\pm} + a^\dagger_- R_{-\pm} \tag{104}$$

where

$$R_{ij} = \cos(\frac{\theta}{2})\delta_{ij} - i\sin(\frac{\theta}{2})(\hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma})_{ij} \tag{105}$$

are elements of a $SU(2)$ matrix R. The binomial theorem gives

$$(a^\dagger_+ R_{+\pm} + a^\dagger_- R_{-\pm})^{n_\pm} = \sum_{k=0}^{n_\pm} \frac{n_\pm!}{k!(n_\pm - k)!} R^k_{+\pm} R^{n_\pm - k}_{-\pm} (a^\dagger_+)^k (a^\dagger_-)^{n_\pm - k}. \tag{106}$$

Using (106) in (95) gives

$$D^i_{\mu\nu}(R(\boldsymbol{\theta}) = \langle n'_+ n'_- | e^{-\frac{i}{2}a^\dagger \hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma} a\theta} | n_+ n_-\rangle =$$

$$\langle 0,0| \frac{a_+^{n'_+}}{\sqrt{n'_+!}} \frac{a_-^{n'_-}}{\sqrt{n'_-!}} \frac{\sum_{k=0}^{n_+} \frac{n_+!}{k!(n_+-k)!} R^k_{++} R^{n_+-k}_{-+} (a^\dagger_+)^k (a^\dagger_-)^{n_+-k}}{\sqrt{n_+!}} \times$$

$$\frac{\sum_{l=0}^{n_-} \frac{n_-!}{l!(n_--l)!} R^l_{+-} R^{n_--l}_{--} (a^\dagger_+)^l (a^\dagger_-)^{n_\pm - l}}{\sqrt{n_-!}} |0,0\rangle = \tag{107}$$

$$\sum_{l=0}^{n_-}\sum_{k=0}^{n_+} \frac{1}{\sqrt{n'_+!}} \frac{1}{\sqrt{n'_-!}} \frac{n_+! n_-!}{k! l! (n_+ - k)! (n_i - l)!} \frac{1}{\sqrt{n_+!}\sqrt{n_-!}}$$

$$\times R^k_{++} R^{n_+-k}_{-+} R^l_{+-} R^{n_--l}_{--} \underbrace{\langle 0,0| a_+^{n'_+} a_-^{n'_-} (a^\dagger_+)^{l+k} (a^\dagger_-)^{n_+ + n_- - k - l}}_{n'_+! n_- l! \delta_{n'_+, l+k} \delta_{n'_, n_+ n_- - l - k}} =$$

$$\sum_{l=0}^{n_-}\sum_{k=0}^{n_+} \frac{\sqrt{n_+!' n_-!' n_+! n_-!}}{k! l! (n^+ - k)! (n_- - l)!} R^k_{++} R^{n_+-k}_{-+} R^l_{+-} R^{n_--l}_{--} \delta_{n'_+, l+k} \delta_{n'_, n_+ n_- - l - k}. \tag{108}$$

Since

$$n'_+ = j+\mu = l+k \qquad n'_- = j-\mu = 2j-l-k \qquad n_+ = j+\nu \qquad n_- = j-\nu \tag{109}$$

it follows that $l = j+\mu-k$ and (108) becomes

$$D^i_{\mu\nu}(R(\boldsymbol{\theta}) =$$

$$\sum_{k=0}^{j+\nu} \frac{\sqrt{(j+\mu)!(j-\mu)!(j+\nu)!(j-\nu)!}}{k!(j+\mu-k)!(j+\nu-k)!(k-\nu-\mu)!} R_{++}^{k} R_{-+}^{j+\nu-k} R_{+-}^{j+\mu-k} R_{--}^{k-\mu-\nu}. \tag{110}$$

Note that this is a homogeneous polynomial of degree $2j$ in the matrix elements of R with real coefficients. Since

$$R_{++} = R^*_{++} \qquad R_{--} = R^*_{--} \qquad R_{+-} = R^*_{-+} \tag{111}$$

$$(D^i_{\mu\nu}(R))^\dagger = (D^i_{\mu\nu}(R^\dagger)) \tag{112}$$

Group integration

If we consider

$$D^j_{\mu\nu}(R) \tag{113}$$

and average this quantity over all possible $SU(2)$ rotations with equal weight the result will be invariant under rotations. Since any $SU(2)$ matrix can be expressed as $R = R'R_0$ where R_0 is any fixed $SU(2)$ matrix, averaging over R or R' will give the same result

$$\int D^j_{\mu\nu}(R)dR = \int D^j_{\mu\nu}(R'R_0)dR =$$

$$\int D^j_{\mu\nu}(R'R_0)dR' = \int D^j_{\mu\alpha}(R')dR' D^j_{\alpha,\nu}(R_0) \tag{114}$$

which is satisfied if

$$D^j_{\mu\alpha}(R') = 0 \text{ or } D^j_{\alpha,\nu}(R_0) = 1. \tag{115}$$

This means

$$\int D^j_{\mu\nu}(R)dR = C\delta_{j0}\delta_{\mu 0}\delta_{\nu 0} \tag{116}$$

where C is a normalization constant. dR can be defined so the constant is 1.

For $R = e^{-\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}$,

$$\boldsymbol{\theta} = \theta(\sin(\chi)\cos(\phi), \sin(\chi)\sin(\phi), \cos(\chi)) \tag{117}$$

and the normalized Haar measure, dR is

$$\boxed{dR = \frac{d\theta \sin(\chi) d\chi d\phi}{32\pi^2}} \tag{118}$$

where (χ, ϕ) are the polar angles on the surface of the unit sphere, representing the axis of rotation, and the angle of rotation and the polar angle ϕ can go from 0 to 4π to cover all angles of rotation in $SU(2)$ (note that a rotation by 2π changes the sign. With this choice of normalization

$$\boxed{\int D^j_{\mu\nu}(R)dR = \delta_{j0}\delta_{\mu 0}\delta_{\nu 0}}. \tag{119}$$

What this equation implies is that there are no non-trivial rotationally invariant subspaces on the subspace spanned by the $|j,\mu\rangle$:

$$|\psi\rangle = \sum_{\mu=-j}^{j} c_\mu |j,\mu\rangle. \tag{120}$$

Because of this $D^j_{\mu\nu}(R)$ is called an $2j+1$ dimensional unitary irreducible representation of $SU(2)$.

Adding angular momentum

Let $\mathbf{J}_a$ and $\mathbf{J}_b$ be angular momentum operators for two independent systems. They satisfy

$$[J_{ai}, J_{aj}] = i\sum_{k=1}^{3} \epsilon_{ijk} J_{ak} \qquad [J_{bi}, J_{bj}] = i\sum_{k=1}^{3} \epsilon_{ijk} J_{bk} \qquad [J_{ai}, J_{bj}] = 0. \tag{121}$$

Define the total angular momentum operator:

$$\mathbf{J}_{ab} = \mathbf{J}_a + \mathbf{J}_b \tag{122}$$

Exercise: Show that

$$[J_{abi}, J_{abj}] = i\sum_{k=1}^{3} \epsilon_{ijk} J_{abk} \tag{123}$$

follows from (121). This means that it is possible to find simultaneous eigenstates of $\mathbf{J}^2_{ab}$ and J_{abz}, $|j_{ab},\mu_{ab}\rangle$. These states can be expressed as linear combinations of the eigenstates of $\mathbf{J}^2_a$, J_{az}, $\mathbf{J}^2_b$ and J_{bz}:

$$|j_{ab},\mu_{ab}\rangle = \sum_{\mu_a=-j_a}^{j_a} \sum_{\mu_b=-j_b}^{j_b} |j_a,\mu_a\rangle \otimes |j_b,\mu_b\rangle C^{j_{ab} j_a, j_b}_{\mu_{ab},\mu_a,\mu_b}. \tag{124}$$

The coefficients $C^{j_{ab} j_a, j_b}_{\mu_{an},\mu_a,\mu_b}$ are called Clebsch Gordan coefficients. This expression only defines them up to phase. We will define a commonly used phase convention in what follows.

By definition

$$\mathbf{J}^2_{ab}|j_{ab},\mu_{ab}\rangle = j_{ab}(j_{ab}+1)|j_{ab},\mu_{ab}\rangle \tag{125}$$

$$J_{abz}|j_{ab},\mu_{ab}\rangle = \mu_{ab}|j_{ab},\mu_{ab}\rangle \tag{126}$$

To proceed note

$$J_{abz} = J_{az} + J_{bz} \tag{127}$$

$$J_{ab\pm} = J_{a\pm} + J_{b\pm} \tag{128}$$

Exercise: Show

$$\mathbf{J}^2_{ab} = J_{ab\pm} J_{ab\mp} + J^2_{abz} \mp J_{abz} \tag{129}$$

In order to build ab states start with

$$|j_a, j_a\rangle \otimes |j_b, j_b\rangle \tag{130}$$

and note

$$J_{abz}|j_a, j_a\rangle \otimes |j_b, j_b\rangle = J_{az}|j_a, j_a\rangle \otimes |j_b, j_b\rangle + J_{bz}|j_a, j_a\rangle \otimes |j_b, j_b\rangle =$$
$$(\mu_a + \mu_b)|j_a, j_a\rangle \otimes |j_b, j_b\rangle \tag{131}$$

and

$$\mathbf{J}_{ab}^2|j_a, j_a\rangle \otimes |j_b, j_b\rangle = (J_{ab-}J_{ab+} + J_{abz}^2 + J_{abz})|j_a, j_a\rangle \otimes |j_b, j_b\rangle =$$
$$j_{ab}(j_{ab} + 1)|j_a j_a\rangle \otimes |j_b, j_b\rangle. \tag{132}$$

It follows that

$$|j_{ab}, j_{ab}\rangle = |j_a, j_a\rangle \otimes |j_b, j_b\rangle N \tag{133}$$

where we can choose the normalization factor $N = 1$. It follows that all of the vectors

$$|j_{ab}, \mu_{ab}\rangle \tag{134}$$

can be constructed by repeated applications of $J_{ab-} = J_{a-} + J_{b-}$ to this state. The expansion coefficients can be read off.

This gives $2j_{ab}+1$ independent vectors, which is less than $(2j_a+1)\times(2j_b+1)$ dimension of the Hilbert space. Since

$$J_{abz}|j_a, \mu_a\rangle \otimes |j_b, \mu_b\rangle = (\mu_a + \mu_b)|j_a, \mu_a\rangle \otimes |j_b, \mu_b\rangle \tag{135}$$

it follows that there is only one vector with eigenvalue of J_{abz} with value j_a+j_b, which must be the state $|j_a, j_a\rangle \otimes |j_b, j_b\rangle$. There are two independent vectors with $\mu_{ab} = j_a + j_b - 1$, $|j_a, j_a - 1\rangle \otimes |j_b, j_b\rangle$ and $|j_a, j_a\rangle \otimes |j_b, j_b - 1\rangle$. One is obtained by applying J_{ab-} to $|j_a, j_a\rangle \otimes |j_b, j_b\rangle$

$$J_-|j_a + j_b, j_a + j_b\rangle =$$
$$\sqrt{2(j_a + j_b)(2(j_a + j_b) + 1)}|j_a + j_b, j_a + j_b - 1\rangle =$$
$$\sqrt{2j_a(2j_a + 1)}|j_a - 1, j_a\rangle \otimes |j_b, j_b\rangle + \sqrt{2j_b(2j_b + 1)}|j_a, j_a\rangle \otimes |j_b, j_b - 1\rangle. \tag{136}$$

The orthogonal complement of this vector vanishes if J_{ab+} is applied to it, since there is only one state with eigenvalue $\mu_a + \mu_b$. This state necessarily has $j_{ab} = j_a + j_b - 1$ (i.e. this is the highest weight). This state can be identified with $|j_a + j_b - 1, j_a + j_b - 1\rangle$ which again is only defined up to normalization by orthogonality

$$|j_a + j_b - 1, j_1 + j_b - 1\rangle =$$
$$N[\sqrt{2j_b(2j_b + 1)}|j_a - 1, j_a\rangle \otimes |j_b, j_b - \sqrt{2j_a(2j_a + 1)}|j_a, j_a\rangle \otimes |j_b, j_b - 1\rangle]. \tag{137}$$

This state has the form

$$|j_a + j_b - 1, j_1 + j_b - 1\rangle =$$

$$|j_a j_a\rangle \otimes |j_b, j_b - 1\rangle C^{j_{ab} j_a j_b}_{j_a+j_b-1, j_a, j_b-1} +$$
$$|j_a j_a - 1\rangle \otimes |j_b, j_b\rangle C^{j_{ab} j_a j_b}_{j_a+j_b-1, j_a-1, j_b-1} \tag{138}$$

The convention is to choose for $j_a \geq j_b$, $C^{j_{ab} j_a j_b}_{j_a+j_b-1, j_a, j_b-1}$ is real and positive and the state is normalized to unity. This is only a convention - it is not required, but it is necessary to stick with one convention in any calculation. Once this state is fixed all of the other states with $j_{ab} = j_a + j_b - 1$ can be obtained using lowering operators

This process can be continued since there are three independent states with $\mu_{ab} = j_a + j_b - 2$. The first two correspond to $j_{ab} = j_a + j_b$ and $j_{ab} = j_a + j_b - 1$. The state orthogonal to these is an eigenstate of J^2_{ab} with eigenvalue $j_a + j_b - 2$. This can be continued until we get the states with $j_{ab} = j_a - j_b$. This process generates all of the Clebsch Gordan coefficients, which are matrix elements of the unitary transformation relating these two bases.
Exercise: Show for $j_a \geq j_b$

$$(2(j_a+j_b)+1)+(2(j_a+j_b-1)+1)+\cdots+(2(j_a-j_b)+1) = (2j_a+1)(2j_b+1) \tag{139}$$

which means that when adding two angular momentum vectors

$$j_a + j_b \geq j_{ab} \geq |j_a - j_b|. \tag{140}$$

and all states are accounted for. The Clebsch Gordan coefficients can be found in tables or generated using computer programs.

Wigner functions:

Since independent angular momentum operators $\mathbf{J}_a$ and $\mathbf{J}_b$ commute it follows that

$$U(R) = e^{-i(\mathbf{J}_a+\mathbf{J}_b)\cdot\boldsymbol{\theta}} = e^{-i\mathbf{J}_a\cdot\boldsymbol{\theta}} e^{-i\mathbf{J}_b\cdot\boldsymbol{\theta}} \tag{141}$$

Taking matrix elements with tensor product states gives a product of Wigner functions

$$\langle j_a, \mu_a| \otimes \langle j_b, \mu_b|U(R)|j_a, \nu_a\rangle \otimes |j_b, \mu_b\rangle =$$
$$\langle j_a, \mu_a|e^{-i\mathbf{J}_a\cdot\boldsymbol{\theta}}|j_a, \nu_a\rangle\langle j_b, \mu_b|e^{-i\mathbf{J}_b\cdot\boldsymbol{\theta}}|j_b, \nu_b\rangle =$$
$$D^{j_a}_{\mu_a\nu_a}(R) D^{j_b}_{\mu_b\nu_b}(R) \tag{142}$$

Using Clebsch Gordan coefficients the tensor product states can be expanded in eigenstates of $\mathbf{J}_a + \mathbf{J}_b$. The left side of (142) can be expressed as

$$\sum_{j=|j_a-j_b|}^{j_a+j_b} \sum_{\mu,\nu=-j}^{j} \langle j_a, \mu_a| \otimes \langle j_b, \mu_b|j, \mu\rangle\langle j, \mu|U(R)|j, \nu\rangle\langle j, \nu|j_a, \nu_a\rangle \otimes |j_b, \mu_b\rangle =$$

$$\sum_{j=|j_a-j_b|}^{j_a+j_b} \sum_{\mu,\nu=-j}^{j} \langle j_a, \mu_a| \otimes \langle j_b, \mu_b|j, \mu\rangle D^j_{\mu\nu}(R)\langle j, \nu|j_a, \nu_a\rangle \otimes |j_b, \mu_b\rangle \tag{143}$$

Comparing these expressions, using the definition of Clebsch Gordan coefficients, (124), gives

$$\boxed{D^{j_a}_{\mu_a\nu_a}(R)D^{j_b}_{\mu_b\nu_b}(R) = \sum_{j=|j_a-j_b|}^{j_a+j_b}\sum_{\mu=-j}^{j} C^{jj_aj_b}_{\mu\mu_a\mu_b}D^j_{\mu\nu}(R)C^{jj_aj_b}_{\nu\nu_a\nu_b}} \tag{144}$$

This decomposes the vector space spanned by the tensor product states into a direct sum of subspaces that transform irreducibly under rotations.

Theorem:

$$\boxed{\int dR D^{j_a}_{\mu_a\nu_a}(R)D^{j_b*}_{\mu_b\nu_b}(R) = \frac{\delta_{j_aj_b}\delta_{\mu_a\mu_b}\delta_{\nu_a\nu_b}}{2j_a+1}} \tag{145}$$

Proof: To prove this note that for $SU(2)$

$$R = e^{-\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}} \tag{146}$$

implies

$$R^* = e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}^*} = e^{-\frac{i}{2}\boldsymbol{\theta}\cdot\sigma_2\boldsymbol{\sigma}\sigma_2} = \sigma_2 R\sigma_2 = (-i\sigma_2)R(i\sigma_2). \tag{147}$$

Using

$$R(\pi\hat{\mathbf{y}}) = \cos(\frac{\pi}{2}) - i\sigma_2\sin(\frac{\pi}{2}) = -i\sigma_2 \tag{148}$$

gives

$$D(R^*) = R(\pi\hat{\mathbf{y}})D(R)R(-\pi\hat{\mathbf{y}}). \tag{149}$$

Since

$$R(\pi\hat{\mathbf{y}}) = -i\sigma_2 = \begin{pmatrix} 0,-1 \\ 1,0 \end{pmatrix} \tag{150}$$

which gives $R_{++} = R_{--} = 0$ and $R_{-+} = -R_{+-} = 1$. Using these in (110) gives

$$D^j_{\mu\nu}(R(\pi\hat{\mathbf{y}})) = \delta_{\mu-\nu}(-)^{j-\mu} \tag{151}$$

and

$$D^j_{\mu,\nu}(R(-\pi\hat{\mathbf{y}})) = \delta_{\mu-\nu}(-)^{j+\mu} \tag{152}$$

Equations (151) and (152) can be used in (149)) to get

$$D^j_{\mu,\nu}(R^*) = \delta_{\mu-\mu'}(-)^{j-\mu}D^j_{\mu',\nu'}(R)\delta_{\nu'-\nu}(-)^{j-\nu} = D^j_{-\mu,-\nu}(R)(-)^{2j-\mu-\nu}$$

$$\int dR D^{j_a*}_{\mu_a,\nu_a}(R)D^{j_b}_{\mu_b,\nu_b}(R) = \int dR D^{j_a}_{-\mu_a,-\nu_a}(R)D^{j_b}_{\mu_b,\nu_b}(R) = (-)^{2j_a-\mu_a-\nu_a}$$

$$(-)^{2j_a-\mu_a-\nu_a}C^{jj_aj_b}_{\mu-\mu_a\mu_b}C^{jj_aj_b}_{\nu-\nu_a\nu_b}\int D^j_{\mu\nu}(R)dR \tag{153}$$

Using (119) gives

$$(116) = (-)^{2j_a-\mu_a-\nu_a}C^{0j_aj_b}_{0-\mu_a\mu_b}C^{0j_aj_b}_{0-\nu_a\nu_b}. \tag{154}$$

This implies

$$j_a = j_b,\ \mu_a = \mu_b,\ \nu_a = \nu_b,\ (-)^{2j_a-\mu_a-\nu_a} = (-)^{2(j_a-\mu_a)} = 1. \tag{155}$$

Using the value of the Clebsch Gordan coefficient

$$C^{0jj}_{0,\mu,\mu} = \frac{(-)^{j-\mu}}{\sqrt{2j+1}} \tag{156}$$

gives

$$\int dR D^{j_a}_{\mu_a\nu_a}(R) D^{j_b *}_{\mu_b\nu_b}(R) = \frac{\delta_{j_a j_b}\delta_{\mu_a\mu_b}\delta_{\nu_a\nu_b}}{2j_a+1} \tag{157}$$

which completes the proof of the theorem.

Tensor Operators

The beginning of the discussion on rotations started by considering how vector operators transform under rotations. There are a number of quantities in physics that transform like products of vectors under rotations. Examples are the inertia tensor from classical mechanics, stress and strain tensors, etc. In general under rotations of the coordinate system the components of these objects transform like

$$V_i' = R_{ij}V_j \tag{158}$$

A general function of an operator with components $\mathbf{x}$ has a Taylor series

$$f(\mathbf{x}) = f(0) + \sum_i f_i x_i + \frac{1}{2} f_{ij} x_i x_j + \cdots \tag{159}$$

Under rotations

$$U^\dagger(R) f(\mathbf{x}) U(R) = f(0) U^\dagger(R) U(R)$$

$$+ \sum_i f_i U^\dagger(R) x_i U(R) + \frac{1}{2} f_{ij} U^\dagger(R) x_i U(R) U^\dagger(R) x_j U(R) + \cdots =$$

$$f(0) + \sum_i f_i R_{ij} x_j + \frac{1}{2} f_{ij} R_{ik} R_{jl} x_j x_l + \tag{160}$$

In this expression first term is invariant, the second transforms like a vector, the third term transform like a products of vectors, etc.

The quantities $x_j x_l$ are components of a rank 2 Cartesian tensor. In general a rank N Cartesian tensor operator has N indices and transforms like

$$\boxed{U^\dagger(R) T^{i_1\cdots i_N} U(R) = \sum R_{i_1 j_1} \cdots R_{i_N j_N} T^{i_1\cdots i_N}.} \tag{161}$$

While Cartesian tensors are useful, they involve non-trivial objects that do not mix under rotations. Clearly the dot product of two vectors transforms like a scalar, the cross product of two vectors transforms like a vector under rotations and the five remaining independent components transform like a traceless symmetric tensor. This means that a general rank two Cartesian tensor is made up of three different types of operators that transform among themselves (scalars to scalars, vectors to vectors, and traceless symmetric tensors to traceless symmetric tensors) under rotations and do not mix.

In general a high rank tensor can be decomposed into parts that transform irreducibly under rotations. A spherical tensor operator is a collection of operators V^i_μ that transform like

$$\boxed{U(R)V^j_\mu U^\dagger(R) = \sum_{\nu=-j}^{j} V^j_\nu D^j_{\nu,\mu}(R)} \tag{162}$$

Theorem: Wigner Eckart - Let V^j_μ be a spherical Tensor operator. Then

$$\boxed{\langle j_a, \mu_a | V^{j_b}_{\mu_b} | \mathrm{j_c}, \mu_c\rangle = C^{j_a j_b j_c}_{\mu_a \mu_b \mu_c} \frac{\langle j_a \| V^{j_b} \| \mathrm{j_c}\rangle}{\sqrt{2j_a+1}}} \tag{163}$$

where $\langle j_a \| V^{j_b} \| \mathrm{j_c}\rangle$ is independent of μ_a, μ_b, μ_c. This means that matrix elements of spherical tensors in $\mathbf{J}^2, J_z$ eigenstates are proportional to Clebsch Gordan coefficients. The quantity $\langle j_a \| V^{j_b} \| \mathrm{j_c}\rangle$ is called the reduced matrix element which is the part that distinguishes different tensor operators with the same j.

proof: To prove this use

$$\langle j_a, \mu_a | V^{j_b}_{\mu_b} | \mathrm{j_c}, \mu_c\rangle = \langle j_a, \mu_a | U^\dagger(R) U(R) V^{j_b}_{\mu_b} U^\dagger(R) U(R) | \mathrm{j_c}, \mu_c\rangle. \tag{164}$$

Inserting intermediate states gives

$$\sum_{\nu_a \nu_b \nu_c} D^{*j_a}_{\mu_a,\nu_a}(R) \langle j_a, \nu_a | V^{j_b}_{\nu_b} | j_c, \nu_c\rangle D^{j_b}_{\nu_b \mu_b}(R) D^{j_c}_{\nu_c \mu_c}(R) =$$

$$\sum_{\nu_a \nu_b \nu_c, j, \nu, \mu} D^{*j_a}_{\mu_a,\nu_a}(R) \langle j_a, \nu_a | V^{j_b}_{\nu_b} | j_c, \nu_c\rangle C^{j \mathrm{j_b} j_c}_{\nu \nu_b \nu_c} D^j_{\nu,\mu}(R) C^{j \mathrm{j_b} j_c}_{\mu \mu_b \mu_c}. \tag{165}$$

Integrating both sides over the Haar measure using (157) gives

$$\sum_{\nu_a \nu_b \nu_c} \langle j_a \nu_a | V^{j_b}_{\nu_b} | j_c \nu_c\rangle \frac{C^{j_a j_b j_c}_{\nu_a \nu_b \nu_c} C^{j_a j_b j_c}_{\mu_a \mu_b \mu_c}}{2j_a+1} \tag{166}$$

which gives

$$\langle j_a \mu_a | V^{j_b}_{\mu_b} | j_c, \mu_c\rangle = C^{j_a j_b j_c}_{\mu_a \mu_b \mu_c} \frac{\langle j_a \| V^{j_b} \| j_c\rangle}{2j_a+1} \tag{167}$$

where

$$\langle j_a \| V^{j_b} \| j_c\rangle = \sum_{\nu_a \nu_b \nu_c} \frac{\langle j_a, \nu_a | V^{j_b}_{\nu_b} | j_c, \nu_c\rangle C^{j_a j_b j_c}_{\nu_a \nu_b \nu_c}}{\sqrt{2j_a+1}} \tag{168}$$

is independent of the magnetic quantum numbers, μ_a, μ_b, μ_c.

This theorem determines selection rules to transitions induced by tensor operators.

Spherical harmonics transform irreducibly under rotations. First note from (162)

$$Y^l_m(\hat{\mathbf{n}}) = \langle \hat{\mathbf{n}}|l,m\rangle = \langle R(\theta,\phi)\hat{\mathbf{z}}|l,m\rangle =$$
$$\langle \hat{\mathbf{z}}|U(R^{-1}(\theta,\phi))|l,m\rangle = \langle \hat{\mathbf{z}}|l,m'\rangle\langle l,m'|U(R^{-1}(\theta,\phi))|l,m\rangle =$$
$$\langle \hat{\mathbf{z}}|l,m'\rangle D^l_{m'm}(R^{-1}(\theta,\phi)) = \langle \hat{\mathbf{z}}|l,m'\rangle D^{*l}_{mm'}(R(\theta,\phi)) \tag{169}$$

where for active rotations

$$R(\theta,\phi) = R_z(\phi)R_y(\theta) \tag{170}$$

and

$$\langle \hat{\mathbf{z}}|l,m'\rangle = \delta_{m'0}\sqrt{\frac{2l+1}{4\pi}} \tag{171}$$

which gives

$$Y^l_m(\hat{\mathbf{n}}) = \sqrt{\frac{2l+1}{4\pi}} D^{*l}_{m0}(R(\theta,\phi)) = \sqrt{\frac{2l+1}{4\pi}} D^{*l}_{m0}(R_z(\phi)R_y(\theta)_y) =$$
$$\sqrt{\frac{2l+1}{4\pi}} e^{im\phi} D^{l*}_{m0}(R_y(\theta)) \tag{172}$$

where $R_z(\phi)R_y(\theta)$ is the rotation that rotates the unit vector with polar angles θ and ϕ to the $\hat{\mathbf{z}}$ axis.

Recall that $l=1$ spherical harmonics are

$$Y^0_1 = \sqrt{\frac{4\pi}{3}}\cos(\theta) = \sqrt{\frac{4\pi}{3}}\frac{z}{r} \tag{173}$$

$$Y^1_1 = -\sqrt{\frac{3}{8\pi}}\sin(\theta)(\cos(\phi) + i\sin(\phi)) = -\sqrt{\frac{3}{4\pi}}\frac{x+iy}{\sqrt{2}r} \tag{174}$$

$$Y^{-1}_1 = \sqrt{\frac{3}{8\pi}}\sin(\theta)(\cos(\phi) - i\sin(\phi)) = \sqrt{\frac{4\pi}{3}}\frac{x-iy}{\sqrt{2}r} \tag{175}$$

Replacing $\mathbf{x}$ by a general vector operator $\mathbf{V}$ we define the spherical components of $\mathbf{V}$ by

$$V^{\pm1}_1 = \mp\frac{V_x \pm iV_y}{\sqrt{2}} \qquad V^0_1 \tag{176}$$

These transform like

$$V^{m\prime}_1 = U(R)V^m_1U^\dagger(R) = \sum_n V^n_1 D^1_{mn}(R) \tag{177}$$

Exercise: Calculate $D^1_{mn}(R)$ for a rotation about the $\hat{\mathbf{z}}$ axis.

Exercise: Express $V_1^{m\prime} = \sum_n V_1^n D^1_{mn}(R)$ in term of cartesian components of $\mathbf{V}$ and show that

$$V^{i\prime} = \sum_j V^j R_{ji} \tag{178}$$

where R_{ij} is an ordinary rotation about the z axis.

Integrating products of spherical harmonics and Wigner functions

The three formulas (119) (145) and (172) can be used to perform an interesting class of integrals.

The steps are as follows

1) Products of Wigner functions can be replaced by a single Wigner function by repeatedly using Clebsch Gordan coefficients

$$\int dR D^{j_1}_{\mu_1\nu_1}[R] D^{j_2}_{\mu_2\nu_2}[R] = \sum_{j\mu\nu} C^{jj_1j_2}_{\mu\mu_1\mu_2} \int dR D^{j_1}_{\mu\nu}[R] C^{jj_1j_2}_{\nu\nu_1\nu_2} =$$

$$C^{0j_1j_2}_{0\mu_1\mu_2} C^{0j_1j_2}_{0\nu_1\nu_2} \tag{179}$$

using (144).

2) Products of Wigner functions and their complex conjugates can be reduced to a single product or a Wigner function and a complex conjugated Wigner function that can be computed using (145)

$$\int dR D^{j_1}_{\mu_1\nu_1}[R] D^{j_2}_{\mu_2\nu_2}[R] D^{j_3*}_{\mu_3\nu_3}[R] = \sum_{j\mu\nu} C^{jj_1j_2}_{\mu\mu_1\mu_2} \int dR D^{j_1}_{\mu\nu}[R] D^{j_3*}_{\mu_3\nu_3}[R] C^{jj_1j_2}_{\nu\nu_1\nu_2} =$$

$$\frac{C^{j_3j_1j_2}_{\mu_3\mu_1\mu_2} C^{j_3j_1j_2}_{\nu_3\nu_1\nu_2}}{2j_3+1} \tag{180}$$

3) Integrals of products of spherical harmonics over polar angles can be expressed as integral of products of Wigner functions over the Haar measure

$$\int \sin\theta d\theta d\phi Y^{m_1}_{l_1}(\theta,\phi) Y^{m_2}_{l_2}(\theta,\phi) =$$

$$\int \sin\theta d\theta d\phi \sqrt{\frac{2l_1+1}{4\pi}} \sqrt{\frac{2l_2+1}{4\pi}} D^{*l_1}_{m_10}(R_z(\phi)R_y(\theta)) D^{*l_2}_{m_20}(R_z(\phi)R_y(\theta))$$

$$\int \sin\theta d\theta d\phi \frac{d\chi}{4\pi} \sqrt{\frac{2l_1+1}{4\pi}} \sqrt{\frac{2l_2+1}{4\pi}} D^{*l_1}_{m_10}(R_z(\phi)R_y(\theta)R_z(\chi)) D^{*l_2}_{m_20}(R_z(\phi)R_y(\theta R_z(\chi))) =$$

$$4\pi \sqrt{\frac{2l_1+1}{4\pi}} \sqrt{\frac{2l_2+1}{4\pi}} \int dR D^{*l_1}_{m_10}(R) D^{*l_2}_{m_20}(R) =$$

$$\sqrt{2l_1+1}\sqrt{2l_2+1} C^{0j_1j_2}_{0\mu_1\mu_2} C^{0j_1j_2}_{000} \tag{181}$$

using (171).