29:5742 final exam
5/7/24

1. Consider a system of three particles in states $\phi_1(\mathbf{r}_1)$, $\phi_2(\mathbf{r}_2)$ and $\phi_3(\mathbf{r}_3)$. Assume that each single particle wave function is normalized to unity and they are mutually orthogonal.

   a. Assume that the three particles are identical Fermions. What is the form of the unit normalized wave function for this three particle system?

   $$\frac{1}{\sqrt{3!}}(\phi_1(\mathbf{r}_1)\phi_2(\mathbf{r}_2)\phi_3(\mathbf{r}_3)+\phi_1(\mathbf{r}_2)\phi_2(\mathbf{r}_3)\phi_3(\mathbf{r}_1)+\phi_1(\mathbf{r}_3)\phi_2(\mathbf{r}_1)\phi_3(\mathbf{r}_2)-$$
   $$\phi_1(\mathbf{r}_2)\phi_2(\mathbf{r}_1)\phi_3(\mathbf{r}_3)-\phi_1(\mathbf{r}_3)\phi_2(\mathbf{r}_2)\phi_3(\mathbf{r}_1)-\phi_1(\mathbf{r}_1)\phi_2(\mathbf{r}_3)\phi_3(\mathbf{r}_2))$$

   b. Assume that the three particles are identical Bosons. What is the form of the unit normalized wave function for this three particle system?

   $$\frac{1}{\sqrt{3!}}(\phi_1(\mathbf{r}_1)\phi_2(\mathbf{r}_2)\phi_3(\mathbf{r}_3)+\phi_1(\mathbf{r}_2)\phi_2(\mathbf{r}_3)\phi_3(\mathbf{r}_1)+\phi_1(\mathbf{r}_3)\phi_2(\mathbf{r}_1)\phi_3(\mathbf{r}_2)+$$
   $$\phi_1(\mathbf{r}_2)\phi_2(\mathbf{r}_1)\phi_3(\mathbf{r}_3)+\phi_1(\mathbf{r}_3)\phi_2(\mathbf{r}_2)\phi_3(\mathbf{r}_1)+\phi_1(\mathbf{r}_1)\phi_2(\mathbf{r}_3)\phi_3(\mathbf{r}_2))$$

   c. Assume that the three particles are distinct Fermions. What is the form of the unit normalized wave function for this three particle system?

   $$\phi_1(\mathbf{r}_1)\phi_2(\mathbf{r}_2)\phi_3(\mathbf{r}_3)$$

2. Consider a Hamiltonian of the form

   $$H = H_0 + \lambda V$$

   where

   $$H_0 = aI + b\sigma_z \qquad V = (I + \sigma_x)$$

   and I is the 2×2 identity, the σ_i are Pauli matrices and a, b are constants and λ is a small constant.

a. What are the eigenvalues and eigenvectors of H_0?

Since

$$H_0 = \begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix}$$

The eigenvalues are $a+b$ and $a-b$ and the eigenvectors are

$$\psi_{a+b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \psi_{a-b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

b. What is the first order correction to the eigenvalues of H due to V?

$$\Delta_{E_+} = \lambda(1,0)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda \qquad \Delta_{E_-} = \lambda(0,1)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

c. What is the second order correction to the eigenvalues of H due to V?

$$\Delta_{E_+}(2) = \frac{|\lambda(1,0)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}|^2}{a+b-a+b} = \frac{\lambda^2}{2b}$$

$$\Delta_{E_-}(2) = \frac{|\lambda(0,1)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}|^2}{a-b-a-b} = -\frac{\lambda^2}{2b}$$

3. A charged particle of charge q and mass m is in a one-dimensional harmonic oscillator well with spring constant k. At time $t = 0$ it experiences an oscillating electric potential

$$\phi(x,t) = -Exe^{-i\lambda t}.$$

Recall for harmonic oscillators $x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$, $\omega = \sqrt{\frac{k}{m}}$, $E_n = \hbar\omega(n + \frac{1}{2})$.

a. Assume the particle is initially in its ground state and the field is weak. Use first order time-dependent perturbation theory to find the probability for a transition to the first excited state at time t.

$$P = |-\frac{i}{\hbar}e^{i\frac{\hbar(\omega(1+\frac{1}{2}-\frac{1}{2}-\lambda)t}{2\hbar}} \frac{\sin(\frac{(\hbar(\omega(1+\frac{1}{2}-\frac{1}{2})-\lambda)t}{2\hbar})}{\frac{(\hbar(\omega(1+\frac{1}{2}-\frac{1}{2})-\lambda)}{2\hbar}} (\frac{-qE}{\sqrt{2\hbar m\omega}}\langle 1|(a+a^\dagger)|0\rangle|^2 =$$

$$\frac{q^2E^2}{2\hbar m\omega} \frac{\sin^2(\frac{(\hbar(\omega-\lambda)t}{2\hbar})}{\frac{(\hbar(\omega-\lambda)}{2}}$$

b. Find the frequency, λ, that maximizes the amplitude this probability.

$\lambda = \omega$ gives the largest contribution

c. Assume that the particle is initially in its ground state and the field is weak. Using first order time-dependent perturbation theory again, what is the probability for a transition to the second excited state at time t?

This vanishes since $\langle 2|(a + a^\dagger)|0\rangle = 0$

4. A many Fermion Hamiltonian has the form

$$H = \sum_n a_n^\dagger \epsilon_n a_n$$

a. Show that this Hamiltonian commutes with the number operator $N = \sum_m a_m^\dagger a_m$.

$$[H, N] = \sum mn\epsilon_m (a_m^\dagger a_m a_n^\dagger a_n - a_n^\dagger a_n a_m^\dagger a_m) =$$

$$\sum_{mn} \epsilon_m (a_m^\dagger (a_m a_n^\dagger + a_n^\dagger a_m - a_n^\dagger a_m) a_n - a_n^\dagger (a_n a_m^\dagger + a_m^\dagger a_n - a_m^\dagger a_n) a_m) =$$

$$\sum_{mn} \epsilon_m (a_m^\dagger (\delta_{mn} - a_n^\dagger a_m) a_n - a_n^\dagger (\delta_{mn} - a_m^\dagger a_n) a_m) =$$

$$\sum_{mn} \epsilon_m (-a_m^\dagger a_n^\dagger a_m a_n - a_n^\dagger a_m^\dagger a_n a_m) + \sum_n \epsilon n (a_n^\dagger a_n - a_n^\dagger a_n) =$$

$$= \sum_{mn} \epsilon_m (-a_m^\dagger a_n^\dagger a_m a_n - (-a_m^\dagger a_n^\dagger)(-a_m a_n)) + \sum_n \epsilon n (a_n^\dagger a_n - a_n^\dagger a_n) = 0$$

b. Show that $|\psi(t)\rangle = a_1^\dagger a_2^\dagger |0\rangle f_{12}$ is an eigenstate of H. What is the eigenvalue of H.

$$H = \sum_n a_n^\dagger \epsilon_n a_n a_1^\dagger a_2^\dagger |0\rangle f_{12} = \sum_n a_n^\dagger \epsilon_n (a_n a_1^\dagger + a_1^\dagger a_n - a_1^\dagger a_n) a_2^\dagger |0\rangle f_{12} =$$

$$\sum_n \epsilon_n (a_n^\dagger \delta_{n1} a_2^\dagger - a_n^\dagger a_1^\dagger (a_n a_2^\dagger + a_2^\dagger a_n - a_2^\dagger a_n)) |0\rangle f_{12} =$$

$$(\epsilon_1 a_1^\dagger a_2^\dagger - \epsilon_2 a_2^\dagger a_1^\dagger) |0\rangle f_{12} = (\epsilon_1 + \epsilon_2) a_1^\dagger a_2^\dagger |0\rangle f_{12}$$

eigenvalue is $(\epsilon_1 + \epsilon_2)$.

c. Write down the Schrödinger equation for the two-particle state $|\psi(t)\rangle = a_1^\dagger a_2^\dagger |0\rangle f_{12}(t)$

$$i\hbar a_1^\dagger a_2^\dagger |0\rangle \frac{df_{12}(t)}{dt} = H a_1^\dagger a_2^\dagger |0\rangle = (\epsilon_1 + \epsilon_2) a_1^\dagger a_2^\dagger |0\rangle f_{12}(t)$$

d. Solve for $f_{12}(t)$ assuming that $f_{12}(0) = f_0$

$$f_{12}(t) = e^{-\frac{i}{\hbar}(\epsilon_1+\epsilon_2)t} f_{12}(0)$$

5. Let V be a potential the form

$$\langle \mathbf{k}|V|\mathbf{k}'\rangle = -\lambda e^{-\alpha(k^2+(k')^2)}$$

a. What is the Born approximation for the transition operator?

$$\langle \mathbf{k}|T(\frac{k'^2}{2\mu} + i\epsilon)|\mathbf{k}'\rangle \approx \langle \mathbf{k}|V|\mathbf{k}'\rangle = -\lambda e^{-2\alpha k^2}$$

since $k^2 + (k')^2)$ by energy conservation

b. What is the center of mass differential cross section in the Born approximation?

$$\frac{d\sigma}{d\Omega(\hat{\mathbf{k}})} = |-(2\pi)^2\mu\hbar|\langle \mathbf{k}|T(\frac{k'^2}{2\mu} + i\epsilon)|\mathbf{k}'\rangle|^2 \approx (2\pi)^4\mu^2\hbar^2\lambda^2 e^{-4\alpha k^2}$$

c. What is the total cross section in the Born approximation. Since the

scattering amplitude is indendent of angles it is enough to multiply by $\int d\Omega(\hat{\mathbf{k}}) = 4\pi$:

$$\sigma_T = 4\pi(2\pi)^4\mu^2\hbar^2\lambda^2 e^{-4\alpha k^2}$$

6. Consider the 2×2 hermitian matrix

$$X = \begin{pmatrix} x^0 + x^2 & x^1 - ix^2 \\ x_1 + ix^2 & x^0 - x_3 \end{pmatrix}$$

and let A be any complex 2×2 matrix with determinant 1 and let $X' = AXA^\dagger$

a. Show A defines a Lorentz transformation.

$$(x'^0)^2 - \mathbf{x}' \cdot \mathbf{x}' = \det(X') = \det(AXA^\dagger) =$$

$$\det(A)\det(X)\det(A^\dagger) = 1 \times \det(X)1^* = \det(X) = (x^0)^2 - \mathbf{x} \cdot \mathbf{x}$$

b. Given A how would you calculate the 4×4 matrix Lorentz transformation.

$$x^{\mu'} = \frac{1}{2}\text{Tr}(\sigma_\mu X') = \frac{1}{2}\text{Tr}(\sigma_\mu A X A^\dagger)\frac{1}{2}\text{Tr}(\sigma_\mu A \sum_\nu x^\nu \sigma_\nu A^\dagger)$$

$$= \sum_\nu \frac{1}{2}\text{Tr}(\sigma_\mu A \sigma_\nu A^\dagger) x^\nu$$

which gives

$$\Lambda^\mu{}_\nu = \frac{1}{2}\text{Tr}(\sigma_\mu A \sigma_\nu A^\dagger)$$

c. Is it possible to find an A that represents space reflection?

Since any A with determinant 1 can be expressed as $A = e^{\mathbf{z}\cdot\boldsymbol{\sigma}}$ it an be continuously deformed to the identity using $A(\lambda) = e^{\lambda\mathbf{z}\cdot\boldsymbol{\sigma}}$ and letting $\lambda : 1 \to 0$. This means that

$$\Lambda^\mu{}_\nu(\lambda) = \frac{1}{2}\text{Tr}(\sigma_\mu A(\lambda)\sigma_\nu A^\dagger(\lambda))$$

has determinant 1, which is not possible for a space reflection.