29:5742 Sample Mid Term Exam

2/22

1. Consider the Hamiltonian

$$H = e \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 4 \\ 0 & 4 & 5 \end{pmatrix}$$

and the trial wave function

$$\psi = \begin{pmatrix} \cos(\phi) \\ 0 \\ \sin(\phi) \end{pmatrix}$$

a. Find the expectation value of the energy as a function of ϕ.

$$\langle\psi|H|\psi\rangle = e(\cos(\phi)^2 + 5\sin^2(\phi)) = e(1 + 4\sin^2(\phi))$$

b. Find the extremal values of the variational parameter ϕ.

$$\frac{d\langle\psi|H|\psi\rangle}{d\phi} = e8\sin(\phi)\cos(\phi) = 4e\sin(2\phi)$$

This vanishes for $\phi \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$

c. Which solutions give an upper bound to the exact lowest energy eigenstate? Compute the bound.

$$\frac{d^2\langle\psi|H|\psi\rangle}{d\phi^2} = 8e\cos(2\phi)$$

This is positive for $\psi \in \{0, \pi\}$

$$E \leq e$$

d. Which solutions give a lower bound to the exact highest energy eigenstate. Compute the bound.

$$\frac{d^2\langle\psi|H|\psi\rangle}{d\psi^2} = 8e\cos(2\phi)$$

This is negative for $\psi \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$

$$E \geq 5e$$

2. $H = \hbar\omega a^\dagger a + i\lambda(a - a^\dagger)$

   a. Treat the term proportional to λ as a perturbation. Use time-independent perturbation theory to find the first-order correction to the ground-state binding energy.

   $$\langle 0|i\lambda(a - a^\dagger)|0\rangle = 0$$

   b. Use time-independent perturbation theory to find the first-order correction to the ground state wave function.

   $$|\psi_0^1\rangle = |1\rangle\frac{\langle 1|i\lambda(a - a^\dagger)|0\rangle}{\hbar\omega(0-1)} = |1\rangle\frac{i\lambda}{\hbar\omega}$$

   c. Use time-independent perturbation theory to find the second-order correction to the ground-state binding energy.

   $$E_0^2 = -\frac{\lambda^2}{\hbar\omega}$$

3. Consider a particle of mass m in a one-dimensional confining potential

   $$V(x) = \lambda\cosh(x)$$

   a. What is the classical momentum for this system in the classically allowed region?

   $$p_{cl} = \pm(2m(E - \lambda\cosh(x)))^{\frac{1}{2}}$$

   b. What are the classical turning points for this system.

   $$\cosh(x) = \frac{E}{\lambda} \qquad x = \pm\cosh^{-1}(\frac{E}{\lambda})$$

   c. How would you determine approximate *ground state* eigenvalue for this system using the WKB method? (DO NOT DO THE INTEGRAL!)

   $$\int_{-\cosh^{-1}(\frac{E}{\lambda})}^{\cosh^{-1}(\frac{E}{\lambda})}(2m(E - \lambda\cosh(x)))^{\frac{1}{2}}dx = (n + \frac{1}{2})\pi$$

4. Consider a two-particle system consisting of a spin 1 and spin 1/2 particle. Let $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_{1/2}$. Let $|J, M\rangle$ be simultaneous eigenstates of J^2 and J_z

a. What are the possible eigenvalues of J^2?

$$\{\frac{3}{4}\hbar^2, \frac{15}{4}\hbar^2\}$$

b. Express

$$|J, M\rangle = |\frac{3}{2}, \frac{3}{2}\rangle$$

as a linear combination of spin 1 and spin 1/2 eigenstates.

$$|\frac{3}{2}, \frac{1}{2}\rangle = |1, 1\rangle|\frac{1}{2}, \frac{1}{2}\rangle$$

c. Express

$$|J, M\rangle = |\frac{3}{2}, \frac{1}{2}\rangle$$

as a linear combination of spin 1 and spin 1/2 eigenstates.

$$J_-|\frac{3}{2}, \frac{3}{2}\rangle = \sqrt{(\frac{3}{2} + \frac{1}{2})(\frac{3}{2} - \frac{1}{2}) + 1)}|\frac{3}{2}\frac{1}{2}\rangle =$$

$$|1, 0\rangle|\frac{1}{2}, \frac{1}{2}\rangle\sqrt{(1+1)(1-1+1)}+$$

$$|1, 1\rangle|\frac{1}{2}, -\frac{1}{2}\rangle\sqrt{(\frac{1}{2} + \frac{1}{2})(\frac{1}{2} - \frac{1}{2} + 1)}$$

$$\downarrow$$

$$|\frac{3}{2}\frac{1}{2}\rangle = \frac{1}{\sqrt{2}}|1, 0\rangle|\frac{1}{2}, \frac{1}{2}\rangle + \frac{1}{2}|1, 1\rangle|\frac{1}{2}, -\frac{1}{2}\rangle$$

d. Given the answer to part *b* explain how you would construct

$$|\frac{1}{2}, \frac{1}{2}\rangle$$

as a linear combination of spin 1 and spin 1/2 eigenstates (do not calculate - just explain).

It must be orthogonal to $|\frac{3}{2}, \frac{1}{2}\rangle$ with the coefficient of $|1, 1\rangle$ positive.