29:5742 First Mid Term Exam
4/22

1. Consider the Hamiltonian

$$H = \begin{pmatrix} E_1 & 0 & \lambda \\ 0 & E_2 & \lambda \\ \lambda & \lambda & E_3 \end{pmatrix}$$

where $E_1 < E_2 < E_3$ and λ is small. Use Rayleigh Schrodinger perturbation theory compute

a. The first order correction to the unperturbed ground state energy eigenvalue.

b. The first order correction to the unperturbed ground state energy eigenvector.

c. The second order correction to the unperturbed ground state energy eigenvalue.

2. Let H be a one dimensional harmonic oscillator Hamiltonian $H = \frac{p^2}{2m} + \frac{kx^2}{2}$.

a. What is the classical momentum?

b. Where are the classical turning points?

c. What is the form of the WKB wave function in the classically forbidden regions ?

d. Write down an the equation for the WKB eigenvalues?

e. Solve for the eigenvalues?

3. The eigenvalues and Hamiltonian for a three dimensional Harmonic oscillator are

$$H = -\frac{\hbar^2}{2m}(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2}) + \frac{1}{2}kr^2 \qquad E_n = \hbar\sqrt{\frac{k}{m}}(2n + l + \frac{3}{2}).$$

a. Use the Hellmann-Feynman theorem to find the expectation value of r^2.

b. Use the Hellmann-Feynman theorem to find the expectation value of p^2.

c. Use the Hellmann-Feynman theorem to find the expectation value of $\frac{1}{r^2}$.

4. Let V_2^μ be a $j = 2$ spherical tensor operator. Consider matrix elements $\langle j_1, \mu_1 | V_2^\mu | j_2, \mu_2 \rangle$

a. What values of j_1 lead to non-zero matrix elements?

b. What values of μ_1 lead to non-zero matrix elements?

c. Assume that $c = \langle j_1, 0|V_2^0|j_2, 0\rangle$ has been measured. What is $\langle j_1, j_1|V_2^0|j_2, j_1\rangle$?

d. Calculate $\int dRU(R)V_2^{\mu}|j_2, \mu_2\rangle$.

express you answers to parts c.) and d.) in terms of Clebsch-Gordan coefficients. Do not evaluate the Clebsch-Gordan coefficients.

$$\int_0^{\pi} \sin^2(\theta)d\theta = \frac{\pi}{2}$$

Solutions

1.

$$H_0 = \begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{pmatrix} \qquad V = \begin{pmatrix} 0 & 0 & \lambda \\ 0 & 0 & \lambda \\ \lambda & \lambda & 0 \end{pmatrix}$$

$$|\psi_1^0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad |\psi_2^0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad |\psi_3^0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

a. The first order correction to the unperturbed ground state energy eigenvalue.

$$E_1^1 = \langle\psi_1^0|V|\psi_1^0\rangle = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \lambda \\ 0 & 0 & \lambda \\ \lambda & \lambda & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$$

b. The first order correction to the unperturbed ground state energy eigenvector.

$$|\psi_1^0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \frac{\begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \lambda \\ 0 & 0 & \lambda \\ \lambda & \lambda & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}{E_1 - E_2} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \frac{\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & \lambda \\ 0 & 0 & \lambda \\ \lambda & \lambda & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}{E_1 - E_3} =$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \frac{\lambda}{E_1 - E_3}$$

c. The second order correction to the unperturbed ground state energy eigenvalue.

$$E_1^2 = \frac{|\langle\psi_3^0|V|\psi_1^0\rangle|^2}{E_1 - E_2} = \frac{|\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & \lambda \\ 0 & 0 & \lambda \\ \lambda & \lambda & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}|^2}{E_1 - E_3} = \frac{\lambda^2}{E_1 - E_3}$$

2.

a. What is the classical momentum?

$$p_{cl} = \pm\sqrt{2m(E - \frac{1}{2}kx^2)}$$

b. Where are the classical turning points?

$$x_\pm = \pm\sqrt{\frac{2E}{k}}$$

c. What is the form of the WKB wave function in the classically forbidden regions ?

$$\psi(x) = \frac{c}{(2m(\frac{1}{2}kx^2) - E)^{1/4}} e^{-\frac{1}{\hbar}\int_{x_+}^{x} \sqrt{(2m(\frac{1}{2}kx'^2)-E)}dx'} \qquad x > x_+$$

$$\psi(x) = \frac{c}{(2m(\frac{1}{2}kx^2) - E)^{1/4}} e^{-\frac{1}{\hbar}\int_{x}^{x_-} \sqrt{(2m(\frac{1}{2}kx'^2)-E)}dx'} \qquad x < x_-$$

d. Write down an the equation for the WKB eigenvalues?

$$\int_{-\sqrt{\frac{2E}{m}}}^{\sqrt{\frac{2E}{m}}} \sqrt{2m(E - \frac{1}{2}kx^2)}dx = \sqrt{2mE}\int_{-\sqrt{\frac{2E}{m}}}^{\sqrt{\frac{2E}{m}}} \sqrt{(1 - \frac{k}{2E}x^2)}dx =$$

$$\sqrt{2mE}\sqrt{\frac{2E}{k}}\int_{-1}^{1}\sqrt{1-u^2}du = 2E\sqrt{\frac{m}{k}}\frac{\pi}{2} = \pi\hbar(n + \frac{1}{2})$$

e. Solve for the eigenvalues?

$$E_n = \hbar\sqrt{\frac{k}{m}}(n + \frac{1}{2})$$

3.

a. Use the Hellmann-Feynman theorem to find the expectation value of r^2.

$$\frac{\partial H}{\partial k} = \frac{r^2}{2}$$

$$\langle\psi|\frac{\partial H}{\partial k}|\psi\rangle = \langle\psi|\frac{r^2}{2}|\psi\rangle = \frac{\partial E}{\partial k} = \frac{\hbar}{2\sqrt{mk}}(2n + l + \frac{3}{2})$$

$$\langle\psi|r^2|\psi\rangle = \frac{\hbar}{\sqrt{mk}}(2n + l + \frac{3}{2})$$

b. Use the Hellmann-Feynman theorem to find the expectation value of p^2.

$$\frac{\partial H}{\partial m} = -\frac{p^2}{2m^2}$$

$$\langle\psi|\frac{\partial H}{\partial k}|\psi\rangle = -\frac{1}{2m^2}\langle\psi|\frac{p^2}{2}|\psi\rangle = \frac{\partial E}{\partial m} = -\frac{\hbar}{2}\sqrt{\frac{k}{m^3}}(2n+l+\frac{3}{2})$$

$$\langle\psi|p^2|\psi\rangle = \hbar\sqrt{mk}(2n+l+\frac{3}{2})$$

c. Use the Hellmann-Feynman theorem to find the expectation value of $\frac{1}{r^2}$.

$$\frac{\partial H}{\partial l} = \frac{\hbar^2(2l+1)}{2mr^2}$$

$$\langle\psi|\frac{\partial H}{\partial l}|\psi\rangle = \frac{\hbar^2(2l+1)}{2m}\langle\psi|\frac{1}{r^2}|\psi\rangle = \frac{\partial E}{\partial l} = \hbar\sqrt{\frac{k}{m}}$$

$$\langle\psi|\frac{1}{r^2}|\psi\rangle = \frac{\sqrt{mk}}{\hbar}\frac{2}{2l+1}$$

4.

a. What values of j_1 lead to non-zero matrix elements?

$$|j_2-2| \le j_1 \le j_2+2$$

b. What values of μ_1 lead to non-zero matrix elements?

$$\mu_1 = \mu + u_2$$

c. Assume that $c = \langle j_1, 0|V_2^0|j_2, 0\rangle$ has been measured. What is $\langle j_1, j_1|V_2^0|j_2, j_1\rangle$?

$$\langle j_1, j_1|V_2^0|j_2, j_1\rangle = c\frac{C^{j_1 2 j_2}_{j_1 0 j_1}}{C^{j_1 2 j_2}_{000}}$$

d. Calculate $\int dRU(R)V_2^\mu|j_2,\mu_2\rangle$.

$$\int dRU(R)V_2^\mu|j_2,\mu_2\rangle = \sum_{\nu,\nu_2}\int dRV_2^\nu|j_2,\nu_2\rangle D^2_{\nu\mu}(R)D^{j_2}_{\nu_2\mu_2}(R)$$

$$\sum_{\nu,\nu_2}\int dRV_2^\nu|j_2,\nu_2\rangle C^{j2j_2}_{\alpha\nu\nu_2}D^j_{\alpha\beta}(R)C^{j2j_2}_{\beta\mu\mu_2} = \sum_{\nu,\nu_2}V_2^\nu|j_2,\nu_2\rangle C^{02j_2}_{0\nu\nu_2}C^{02j_2}_{0\mu\mu_2} =$$

$$\sum_{\nu=-2}^{2} V_2^{\nu} |2,-\nu\rangle C^{022}_{0\nu-\nu} C^{022}_{0\mu-\mu} \delta_{\mu_2,-\mu} \delta_{j_2,2}$$