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1. Assume $H = H_0 + V$. In the interaction picture the time evolution operator is
$$U_I(t,t') = e^{iH_0t/\hbar}e^{-iH(t-t')/\hbar}e^{-iH_0t/\hbar}$$
which satisfies
$$\frac{dU_I(t,t')}{dt} = -\frac{i}{\hbar}V_I(t)U_I(t,t')$$
$$U_I(t,t') = I - \frac{i}{\hbar}\int_{t'}^{t} V_I(t'')U_I(t'',t')dt''$$
where
$$V_I(t) = e^{iH_0t/\hbar}V_s e^{-iH_0t/\hbar}$$

   a. Write down the Dyson series for the scattering operator, S.

   b. Use this to calculate the transition matrix elements in the Born approximation. Assume V_s is translationally invariant.

   c. Use the result of part b.) to find the center of mass differential cross section in the Born approximation.

2. The scattering amplitude and partial wave phase shifts are related by:
$$F(\mathbf{k}',\mathbf{k}) = \sum_{l=0}^{\infty}\frac{(2l+1)\hbar}{k}e^{i\delta_l}\sin(\delta_l)P_l(\hat{\mathbf{k}}'\cdot\hat{\mathbf{k}})$$

   a. Express the center of mass *differential* cross section as a function of the phase shifts.

   b. Use the result of part a) to calculate the total cross section as a function of the phase shifts.

   c. What is the maximum contribution of the $l-th$ partial wave to the total cross section?

   d. Assume that only the three lowest partial waves, $l = 0,1,2$, have non-zero phase shifts. What is the maximum size of the total cross section.

3. The wave operators used to formulate the scattering asymptotic condition $\Omega_\pm$ are defined by
$$\Omega_\pm \lim_{t\to\pm\infty} e^{iHt/\hbar}e^{-iH_0t/\hbar}$$

   a. Show $e^{iHs/\hbar}\Omega_\pm = \Omega_\pm e^{iH_0s/\hbar}$

   b. Show $H\Omega_\pm = \Omega_\pm H_0$

c. Use the result of part *b.*) to show that the scattering operator $S := \Omega_+^\dagger \Omega_-$ conserves energy.

d. Show that if $|B\rangle$ is a bound state of H with energy eigenvalue $E_B < 0$ that

$$\langle B|\Omega_\pm|\psi\rangle = 0$$

4. Consider the scattering of two nucleons of mass m interacting via an attractive Yukawa interaction of the form

$$V(r) = -\lambda \frac{e^{-\alpha r}}{r}.$$

a. Find the scattering amplitude for this reaction in the first Born approximation.

b. Find the differential cross section (in the Born approximation) as a function of angle in the center of momentum frame.

c. Use the result of part b.) to find the differential cross section (in the Born approximation) for electron-positron scattering with a Coulomb potential.

d. How does the result of part c.) change for electron-electron scattering.

$$\int_1^2 P_l(u) P_{l'}(u) du = \frac{2}{2l+1} \delta_{ll'}$$

$$\cos^2(\frac{\theta}{2}) = \frac{1+\cos(\theta)}{2} \qquad \sin^2(\frac{\theta}{2}) = \frac{1-\cos(\theta)}{2}$$

$$\sum_{m=-l}^{l} Y_m^l(\hat{\mathbf{k}}') Y_m^{l*}(\hat{\mathbf{k}}) = \frac{2l+1}{4\pi} P_l(\hat{\mathbf{k}}' \cdot \hat{\mathbf{k}}) \qquad P_l(1) = 1$$

#1

a) $$S = I + \sum_{n=1}^{\infty} \frac{1}{n!}\left(-\frac{i}{\hbar}\right)^n \int_{-\infty}^{\infty} T(V_I(t_1) \quad V_I(t_n))\, dt_1 \quad dt_n$$

b) $$\langle \bar{k}' | S | k \rangle = \delta(\bar{k}' - \bar{k}) +$$

$$-\frac{i}{\hbar}\int_{-\infty}^{\infty} e^{(i E(k')/\hbar - i E(k)/\hbar)t} \langle \bar{k}' | V | \bar{k} \rangle + \cdots$$

$$\delta(\bar{k}' - \bar{k}) - \frac{i}{\hbar}\, 2\pi\, \delta\left(\frac{E' - E}{\hbar}\right) \langle \bar{k}' | V | k \rangle + \cdots$$

$$\delta(\bar{k}' - \bar{k}) - 2\pi i\, \delta(E' - E) \langle \bar{k}' | V | \bar{k} \rangle$$

$$\langle \bar{k}' | T | \bar{k} \rangle \simeq \langle \bar{k}' | V | \bar{k} \rangle$$

c) $$\frac{d\sigma}{d\Omega} = |F|^2 = (2\pi)^4 \hbar^2 \mu^2 |\langle \bar{k}' | V | k \rangle|^2$$

#2

a) $\frac{d\sigma}{d\Omega} = |F|^2 =$

$$\sum_{l,l'} \frac{(2l+1)}{k}\frac{(2l'+1)}{k}\hbar^2 e^{i(\delta_l - \delta_l')} \sin\delta_l \sin\delta_l' \times$$

$$P_l(\hat{k}'\cdot\hat{k})\, P_l'(\hat{k}'\cdot\hat{k})$$

b) Let $u = \hat{k}'\cdot\hat{k}$ $\quad \int d\Omega = \int d\phi\, du$

$$\int P_l(u) P_l'(u)\, du = \frac{2}{2l+1}\delta_{ll'} \implies$$

$$\sum 2\cdot 2\pi \frac{2l+1}{k^2}\hbar^2 \sin^2\delta_l =$$

$$\frac{4\pi\hbar^2}{k^2}\sum (2l+1)\sin^2\delta_l$$

c) $\frac{4\pi}{k^2}(2l+1)\hbar^2$

d) $\frac{4\pi\hbar^2}{k^2}(1+3+5) = \frac{36\pi}{k^2}\hbar^2$

#3

a) $\lim e^{iHs} e^{iHt} e^{-iH_0 t} =$

$\lim e^{iH(s+t)} e^{-iH_0(s+t)} e^{iH_0 s}$

let $t' = s+t \Rightarrow$

$e^{iHs} \Omega_\pm = \Omega_\pm e^{iH_0 s}$

b) differentiate $\Rightarrow$

$H \Omega = \Omega H_0$

c) $H_0 S = H_0 \Omega_+^\dagger \Omega_- = \Omega_+^\dagger H \Omega_- = \Omega_+^\dagger \Omega_- H_0 = S H_0$

$\therefore [H_0, S] = 0$

which means S conserves energy

d) $0 = \langle b | \Psi_\pm \rangle = \langle b | \Omega_\pm | \Psi^0_\pm \rangle$

since this holds for any $|\Psi^0_\pm\rangle$

$\langle b | \Omega_\pm = 0$

#4

a) $\langle \bar{k}' | V | \bar{k} \rangle =$

$$\frac{1}{(2\pi\hbar)^3} \int e^{-i(\bar{k}'-\bar{k})\cdot\bar{r}/\hbar} \left(-\lambda \frac{e^{-\alpha r}}{r}\right) r^2\, dr\, d\Omega =$$

$$\frac{2\pi}{(2\pi\hbar)^3} \int e^{-i|k-k'|ur/\hbar} \left(-\lambda e^{-\alpha r} r\, dr\right) du$$

$$\frac{2\pi}{(2\pi\hbar)^3} \frac{-\lambda\hbar}{-i|k-k'|} \int e^{-\alpha r} \left(e^{-i|k-k'|r/\hbar} - e^{+i|k-k'|r/\hbar}\right) =$$

$$+\frac{2\pi\lambda\hbar\,(-1)}{(2\pi\hbar)^3\,(i|k-k'|} \left[\frac{1}{-\alpha - i|k-k'|/\hbar} - \frac{1}{-\alpha + i|k-k'|/\hbar}\right]$$

$$\frac{2\pi\lambda\hbar\,(-)}{(2\pi\hbar)^3\,(i|k-k'|} \left[\frac{-\alpha + i|k-k'|/\hbar + \alpha + i|k-k'|/\hbar}{\alpha^2 + |k-k'|^2/\hbar^2}\right]$$

$$-\frac{4\pi\lambda}{(2\pi\hbar)^3} \frac{1}{\alpha^2 + |k-k'|^2/\hbar^2}$$

$$F = -(2\pi)^2 u\hbar \left(-\frac{4\pi\lambda}{(2\pi\hbar)^3} \frac{1}{\alpha^2 + |k-k'|^2/\hbar^2}\right)$$

b) $$\frac{d\sigma}{d\Omega} = |F|^2 = \left(\frac{2u\lambda}{\hbar^2}\right)^2 \left(\frac{\hbar^2}{\alpha^2\hbar^2 + 2k^2(1-\cos\theta)}\right)^2$$

c) let $\alpha = 0$ $\lambda = -e^2$

$$\frac{d\sigma}{d\Omega} = |F|^2 \quad \frac{4u^2e^4}{4k^4(1-\cos\theta)^2}$$

d) no change because of e^4