Decomposition into direct sum of irreducible representations

$$\langle j_1 \mu_1 j_2 \mu_2 | e^{-i(\bar{J}_1+\bar{J}_2)\cdot\bar{\Theta}} | j_1 \nu_1 j_2 \nu_2 \rangle =$$

$$\langle j_1 \mu_1 | e^{-i\bar{J}_1\cdot\bar{\Theta}} | j_1 \nu_1 \rangle \langle j_2 \mu_2 | e^{-i\bar{J}_2\cdot\bar{\Theta}} | j_2 \nu_2 \rangle =$$

$$D^{j_1}_{\mu_1\nu_1}(\bar{\Theta})\, D^{j_2}_{\mu_2\nu_2}(\bar{\Theta}) =$$

$$\sum \langle j_1\mu_1 j_2\mu_2 | j\mu \rangle \langle j\mu | e^{-i(\bar{J}_1+\bar{J}_2)\cdot\bar{\Theta}} | j\nu \rangle \langle j\nu | j_1\nu_1 j_2\nu_2 \rangle =$$

$$\sum C^{j\, j_1\, j_2}_{\mu\, \mu_1\, \mu_2}\, D^{j}_{\mu\nu}(\bar{\Theta})\, C^{j\, j_1\, j_2}_{\nu\, \nu_1\, \nu_2}$$

* subspaces for differen values of j do not

i.e. if a vector sta

out as a linear c ination

of $|j\mu\rangle$ states it s

in that subspace des

rotations

$$C D^{j_1} \otimes D^{j_2} C^{+} = \begin{pmatrix} [\,|j_1-j_2|\,] & 0 & & \\ 0 & [\,|j_1-j_2|+1\,] & & \\ & & \ddots & \\ & & & (j_1+j_2) \end{pmatrix}$$

The Clebsch Gordan Coefficients block diagonalize the rotation matrix into invariant subspaces

* using Clebsch Gordan Coeff. to integrate arbitrary products of Wigner D-matrices

use $\int dR \, D^{j}_{\mu\nu}(R) = \delta_{j0}\delta_{\mu 0}\delta_{\nu 0}$

consider

$$\int dR \, D^{j_1}_{\mu_1\nu_1}(R) \, D^{j_2}_{\mu_2\nu_2}(R) =$$

$$\int \sum C^{j \; j_1 \; j_2}_{\mu \; \mu_1 \; \mu_2} D^{j}_{\mu\nu}(R) \, C^{j \; j_1 \; j_2}_{\nu \; \nu_1 \; \nu_2} \, dR =$$

$$C^{0 \; j_1 \; j_2}_{0 \; \mu_1 \; \mu_2} \; C^{0 \; j_1 \; j_2}_{0 \; \nu_1 \; \nu_2}$$

It follows from properties of the clebsch Gordan coefficients that this integral vanishes unless

$j_1 = j_2 \quad \mu_1 = -\mu_2 \quad \nu_1 = -\nu_2$

This can be generalized to arbitrary products using successive couplings.

Theorem

$$\boxed{\int D^{j_1}_{\mu_1\nu_1}(R)\, D^{2}_{\mu_2\nu_2}(R)^{*}\, dR = \frac{\delta_{j_1 j_2}\,\delta_{\mu_1 \nu_2}\,\delta_{\nu_1\nu_2}}{2j_1+1}}$$

This can be used to integrate arbitrary products of $D^{j}_{\mu\nu}(R)$'s and $D^{j\,*}_{\mu\nu}(R)$'s

Proof of Theorem

$$R = e^{-\frac{i}{2}\bar{\theta}\cdot\bar{\sigma}}$$

$$R^{*} = e^{\frac{i}{2}\bar{\theta}\cdot\bar{\sigma}^{*}} = e^{-\frac{i}{2}\bar{\theta}\cdot(\sigma_2\bar{\sigma}\sigma_2)}$$

$$= \sigma_2 R \sigma_2$$

next note that

$$e^{-i\frac{\pi}{2}\sigma_y} = \cos(\tfrac{\pi}{2}) I - i\sigma_2 \sin(\tfrac{\pi}{2})$$

$$= -i\sigma_2$$

$$R_y(\tfrac{\pi}{2}) = -i\sigma_2 \qquad R_y^{-1}(\tfrac{\pi}{2}) = i\sigma_2$$

this implies

$$R^* = R_y(\tfrac{\pi}{2})\, R\, R_y^{-1}(\tfrac{\pi}{2})$$

$$\therefore \int D^{j_1}_{\mu_1\nu_1}(R)\, D^{j_2 *}_{\mu_2\nu_2}(R)\, dR =$$

$$\int D^{j_1}_{\mu_1\nu_1}(R) \sum D^{j_2}_{\mu_2\alpha}(R_y(\tfrac{\pi}{2}))\, D^{j_2}_{\alpha\beta}(R)\, D^{j_2}_{\beta\nu_2}(R_y^{+}(\tfrac{\pi}{2})) =$$

$$\int\sum C^{j\, j_1\, j_2}_{\mu\, \mu_1\, \alpha}\, C^{j\, j_1\, j_2}_{\nu\, \nu_1\, \beta}\, D^{j}_{\mu\nu}(R)\, D^{j_2}_{\mu_2\alpha}(R_y(\tfrac{\pi}{2}))\, D^{j_2}_{\beta\nu_2}(R_y^{+}(\tfrac{\pi}{2})$$

$$= \sum C^{0\, j_1\, j_2}_{0\, \mu_1\, \alpha}\, C^{0\, j_1\, j_2}_{0\, \nu_1\, \beta}\, D^{j_2}_{\mu_2\alpha}(R_y(\tfrac{\pi}{2}))\, D^{j_2}_{\beta\nu_2}(R_y^{+}(\tfrac{\pi}{2}))$$

$$= C^{0\, j_1\, j_2}_{0\, \mu_1\, -\mu_1}\, C^{0\, j_1\, j_2}_{0\, \nu_1\, -\nu_1}\, D^{j_2}_{\mu_2 -\mu_1}(R_y(\tfrac{\pi}{2}))\, D^{j_2}_{-\nu_1\nu_2}(R_y^{+}(\tfrac{\pi}{2}))$$

note

$$R_y(\tfrac{\pi}{2}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad R_{++} = R_{--} = 0$$

$$R_{+-} = -R_{-+} = -1$$

These can be evaluated using the explicit expression that we derived for $D^{j}_{\mu\nu}(R)$ in terms of the matrix elements of R.

Combining this with the value of the Clebsch Gordan coefficients gives the result

$$\int D^{j_1}_{\mu_1\nu_1}(R)\, D^{j_2 *}_{\mu_2\nu_2}(R)\, dR$$

$$= \frac{\delta_{j_1 j_2}\,\delta_{\mu_1\mu_2}\,\delta_{\nu_1\nu_2}}{2j_1+1}$$

An arbitrary product of D(R)'s and D(R)*'s can be reduced to this integral using Clebsch Gordan coefficients

We can also use this to compute integrals involving spherical harmonics. The trick is to express the spherical harmonics in terms of the Wigner functi-

$$Y_\ell^m(\theta\phi) = \langle \hat{r} | \ell m \rangle$$

where $\hat{r}$ is the unit vector

$$\hat{r} = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)$$

$$\hat{r} = R_z(\phi) R_y(\theta) \hat{z} =$$

$$R_z(\phi) R_y(\theta) \underbrace{R_z(\psi)}_{\text{leaves } \hat{z} \text{ unchanged}} \hat{z}$$

$$= R\hat{z}$$

$$\therefore Y_\ell^m(\theta\phi) = \langle R\hat{z} | \ell m \rangle =$$

$$= \langle \hat{z} | u^\dagger(R) | \ell m \rangle =$$

$$\sum_{m'} \langle z | \ell m' \rangle \langle \ell m' | U^\dagger(R) | \ell m \rangle =$$

$$\sum_{m'} \langle \hat{z} | \ell m' \rangle \langle \ell m | U(R) | \ell m' \rangle^* =$$

$$\sum_{m'} \langle \hat{z} | \ell m' \rangle \; D^{\ell *}_{m m'}(R)$$

but $\langle \hat{z} | \ell m' \rangle = 0$ unless $m' = 0$

$$= \langle \hat{z} | \ell 0 \rangle \, D^*_{m0}(R)$$

$$= Y_\ell^0(0,0) \, D^*_{m0}(R)$$

where you can look up

$$Y_\ell^0(00) = \sqrt{\frac{2\ell+1}{4\pi}}$$

which shows that the spherical harmonics are proportional to the wigner D matrices

$$Y_\ell^m(\hat{r}) = \sqrt{\frac{2\ell+1}{4\pi}} \, D^*_{m0}(R)$$

we can use this identity to calculated integrals over arbitrary products of spherical harmonics, wigner D functions and their complex conjugates.

example:

$$\int \sin\theta\, d\theta\, d\phi\; Y_{\ell_1}^{m_1}(\theta\phi)\; Y_{\ell_2}^{m_2}(\theta\phi) =$$

$$\int \sin\theta\, d\theta\, d\psi \sqrt{\frac{2\ell_1+1}{4\pi}}\sqrt{\frac{2\ell_2+1}{4\pi}} =$$

$$D^{\ell_1 *}_{m_1 0}(R)\, D^{\ell_1 *}_{m_2 0}(R)\, \frac{1}{\underbrace{2\times 4\pi}}\; \frac{32\pi^2}{32\pi^2}$$

do the θ integral twice and the φ integral

$$\int \sqrt{\frac{2\ell_1+1}{4\pi}}\sqrt{\frac{2\ell_2+1}{4\pi}}\, D^{\ell_1 *}_{m_1 0}(R)\, D^{\ell_2 *}_{m_2 0}(R)\, dR \times 4\pi =$$

$$\sqrt{2\ell_1+1}\sqrt{2\ell_2+1}\; C^{J\,\ell_1\ell_2}_{\mu\, m_1 m_2}\; C^{J\,\ell_1\ell_2}_{\nu\,0\,0}\; D^{*\ell}_{\mu\nu}(R)\, dR$$

$$\sqrt{2\ell_1+1}\sqrt{2\ell_2+1}\; C^{0\,\ell_1\ell_2}_{0\,m_1 m_2}\; C^{0\,\ell_1\ell_1}_{0\,0\,0} =$$

$$(2\ell_1+1)\times\delta_{\ell_1\ell_2}\; C^{0\,\ell_1\ell_1}_{0\,m_1 -m_1}\; C^{0\,\ell_1\ell_1}_{0\,0\,0}\; \delta_{m_1 -m_2}$$

obviously this can be extended to treat more complicated integrals

Spherical Tensors and the Wigner Eckart Theorem

Cartesian tensors are quantities that transform like products of vectors under rotations

$$X_a^i X_b^j$$

$$\mathcal{U}(R) X_a^i X_b^j \mathcal{U}^\dagger(R) =$$

$$\mathcal{U}(R) X_a^i \mathcal{U}^\dagger(R) \mathcal{U}(R) X_b^j \mathcal{U}^\dagger(R) =$$

$$\sum_{kl} X_a^k X_b^l R^{ki} R^{lj}$$

This can be abstracted to a rank N cartesian tensor operator

$$V^{i_1 \cdots i_N}$$

That transforms like

$$\mathcal{U}(R) V^{i_1 \cdots i_n} \mathcal{U}^\dagger(R) =$$

$$\sum_{j_1 \, j_n} V^{j_1 \; j_n} R_{i_1 j_1} R_{i_2 j_2} \cdots R_{i_n j_n}$$

example

$$f(\vec{r}) = f(\vec{0}) + \sum \underbrace{\frac{1}{n!} \frac{\partial^n f}{\partial x_{i_1} \cdots x_{i_n}}}_{\text{constants}}(0) \; \underbrace{x_{i_1}^{i_1} \cdots x_{i_n}^{i_n}}_{\text{products of vector}}$$

multipole operators

$F^{\mu\nu}$ electromagnetic field strength tensor

$T^{\mu\nu}$ stress energy tensor

Consider $x^i y^j$

$\bar{x} \cdot \bar{y}$ transforms like a scalar

$\bar{x} \times \bar{y}$ transforms like a vector

The five remaining matrix elements transform

Like a traceless symmetric tensor (trace is $\vec{x}\cdot\vec{y}$, cross product is antisymmetric)

remark - recall spherical harmonics with $\ell=1$

$$Y_1^0(\theta\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\frac{z}{r}$$

$$Y_1^1(\theta\phi) = -\sqrt{\frac{3}{8\pi}}\sin\theta(\cos\phi + i\sin\phi)$$

$$Y_1^{-1}(\theta\phi) = \sqrt{\frac{3}{8\pi}}\sin\theta(\cos\phi - i\sin\phi)$$

$$Y_1^1 = -\sqrt{\frac{3}{4\pi}}\left(\frac{x+iy}{\sqrt{2}}\right)$$

$$Y_{-1}^1 = \sqrt{\frac{3}{4\pi}}\left(\frac{x-iy}{\sqrt{2}}\right)$$

$$x, y, z \longrightarrow -\frac{x+iy}{\sqrt{2}} \quad \frac{x-iy}{\sqrt{2}} \quad z$$

are called the spherical components of $\vec{r}$.

If we think of $Y_1^m(\theta\phi)$ as an operator

$$\mathcal{U}(R)\,Y_1^m(\theta\phi)\,\mathcal{U}^\dagger(R) = \Sigma\, Y_1^{m'}(\theta\phi)\,D^1_{m'm}(R)$$

note

$$\mathcal{U}(R)\;X^m\;{}^{+}(R) = \Sigma\, X^e R_{em}$$

This is generalized to a spherical tensor operator

$$\mathcal{U}(R)\, V_e^m\, \mathcal{U}^\dagger(R) = \sum_{m'} V_e^{m'}\, D^{e}_{m'm}(R)$$

Cartesian tensors transform like products of R, while spherical tensors transform irreducibly

The Wigner Eckart Theorem

- $*$ Let V_e^m be a spherical tensor operator
- $*$ $\langle j\mu | V_e^m | j'\nu' \rangle =$

$$C^{j\ \ell\ j'}_{\mu\ m\ \nu'} \times \text{Constant}\,(j,\ell,j')$$

The implication is that the matrix element are proportional to a Clebsch Gordon coefficient.

note

$$\frac{\langle j_a \mu_a | V_{j_b}^{\mu_b} | j_c \mu_c \rangle}{\langle j \nu_a | V_{j_b}^{\nu_b} | j_c \nu_c \rangle} = \frac{C^{j_a\, j_b\, j_c}_{\mu_a\, \mu_b\, \mu_c}}{C^{j_a\, j_b\, j_c}_{\nu_a\, \nu_b\, \nu_c}}$$

This means that if we know the matrix element – then we can find all of the others.

Proof:

$$\langle j_a \mu_a | V_{j_b}^{\mu_b} | j_c \mu_c \rangle =$$

$$\langle j_a \mu_a | \mathcal{U}^\dagger(R)\, \mathcal{U}(R)\, V_{j_b}^{\mu_b}\, \mathcal{U}^\dagger(R)\, \mathcal{U}(R) | j_c \mu_c \rangle =$$

$$\int dR \, \langle j_a \mu_a | \mathcal{U}^\dagger(R)\, \mathcal{U}(R)\, V_{j_b}^{\mu_b}\, \mathcal{U}^\dagger(R)\, \mathcal{U}(R) | j_c \mu_c \rangle =$$

$$\int dR \; D^{*j_a}_{\nu_a \mu_a}(R) \langle j_a \nu_a | V_{j_b}^{\nu_b} | j_c \nu_c \rangle$$

$$\times\, D^{j_b}_{\nu_b \mu_b}(R)\, D^{j_c}_{\nu_c \mu_c}(R) =$$

$$\int dR \sum D^{*\,j_a}_{\gamma_a \mu_-}(R) \langle j_a \nu_a | V^{j_b}_{\nu_b} | j_c \nu_c \rangle$$

$$1\, j_b\, j_c \quad C^{j \; j_b \; j_c}_{\mu \; \mu_b \mu_c} \; D^{j}_{\gamma\mu}(R) =$$

$$\frac{\delta_{\mu\mu_a} \delta_{\gamma\nu_a} \delta_{jj_a}}{2j_a+1} \; C^{j_a \; j_b \; j_c}_{\mu_a \mu_b \mu_c} \underbrace{\sum_{\nu_a \nu_b \nu_c} C^{j_a \; j_b \; c}_{\nu_a \nu_b \nu_c} \langle j_a \nu_a | V^{j_b}_{\nu_b} | j_c \nu_c \rangle}_{\text{independent of } \mu_a \mu_b \mu_c}$$

$$= C^{j_a \; j_b \; j_c}_{\mu_a \mu_b \mu_c} \times \text{const}$$

$$\text{const} \quad \sum \frac{C^{j_a \; j_b \; j_c}_{\nu_a \nu_b \nu_c} \langle j_a \nu_a | V^{\nu_b}_{j_b} | j_c \nu_c \rangle}{2j_a+1}$$

This completes the proof of the Wigner Eckart theorem

*This is used to derive selection rules.