Lecture 5

adding angular momentum

$[\bar{J}_a, \bar{J}_b] = 0$

define

$\vec{J}_{ab} = \bar{J}_a + \bar{J}_b$

homework

$[J^i_{ab}\ J^j_{ab}] = i \sum_{k=1}^{3} \varepsilon_{ijk} \bar{J}^k_{ab}$

this means that we can find simultaneous eigenstates of $\bar{J}^2_{ab}$ and J^z_{ab}

$|j_{ab}\, m_{ab}\rangle$

the problem is to find these eigenstates as a linear combination of the states

$|j_a\, m_a\, j_b\, m_b\rangle$

$$|j_{ab}\, m_{ab}\rangle = \sum_{m_a m_b} |j_a\, m_a\, j_b\, m_b\rangle \langle j_a\, m_a\, j_b\, m_b | j_{ab}\, m_{ab}\rangle$$

where the coefficients

$$\langle j_a m_a j_b m_b | j_{ab}\, m_{ab}\rangle \equiv C^{j_{ab}\, j_a\, j_b}_{m_{ab}\, m_a\, m_b}$$

are called Clebsch Gordan coefficients

For the construction we use

$$J_\mp J_\pm = \vec{J}^2 - J_z^2 \mp J_z = J^2 - J_z(J_z \pm 1)$$

this can be solved for

$$J^2 = J_- J_+ + J_z(J_z+1)$$

we start by considering

$$J^z_{ab} \, | j_a j_a \, j_b j_b \rangle = (J_a^z + J_b^z) | j_a j_a \, j_b j_b \rangle =$$

$$(j_a + j_b) | j_a j_a \, j_b j_b \rangle$$

$$J^2_{ab} | j_a j_a \, j_b j_b \rangle = (J_{ab-} J_{ab+} + J_z(J_z+1)) | j_a j_a \, j_b j_b \rangle$$

$$= (0 + (j_a + j_b)(j_a + j_b + 1)) \, | j_a j_a \, j_b j_b \rangle$$

since

$$J_{ab+} | j_a j_a \, j_b j_b \rangle = (J_{a+} + J_{b+}) | j_a j_a \, j_b j_b \rangle = 0$$

* This means that $| j_a j_a \, j_b j_b \rangle$ is an eigenstate of J^z_{ab} $\vec{J}^2_{ab}$ with eigenvalues $j_{ab}(j_{ab}+1)$ $j_{ab} = j_a + j_b$

* it is also a highest weight state since

$$J_{+ab} |j_a j_a j_b j_b\rangle = 0$$

we define:

$$\boxed{|j_a + j_b, j_a + j_b\rangle \equiv |j_a j_a j_b j_b\rangle}$$

Recall as a consequence of the commutation relations

$$J_\pm |j\mu\rangle = |j\,\mu\pm 1\rangle \sqrt{(j\mp\mu)(j\pm\mu+1)}$$

using $J^-_{ab} = J^-_a + J^-_b$ on $|j_a + j_b\ j_a + j_b\rangle$ gives

$$J_- |j_a + j_b, j_a + j_b\rangle =$$

$$|j_a + j_b, j_a + j_b - 1\rangle \sqrt{2(j_a + j_b)\cdot 1} =$$

$$(J_{a-} + J_{b-}) |j_a j_a j_b j_b\rangle =$$

$$|j_a j_a j_b j_b - 1\rangle \sqrt{2 j_b} + |j_a j_a - 1, j_b j_b\rangle \sqrt{2 j_a}$$

comparing both sides of this equation gives

$$|j_a+j_b, j_a+j_b-1\rangle = \sqrt{\frac{j_b}{j_a+j_b}}\,|j_a\, j_a\, j_b\, j_b-1\rangle + \sqrt{\frac{j_a}{j_a+j_b}}\,|j_a\, j_a-1\, j_b\, j_b\rangle$$

this means

$$\boxed{\begin{aligned} C^{j_a+j_b\ \ j_a\ \ j_b}_{j_a+j_b-1\ \ j_a\ \ j_b-1} &= \sqrt{\frac{j_b}{j_a+j_b}} \\ C^{j_a+j_b\ \ j_a\ \ j_b}_{j_a+j_b-1\ \ j_a-1\ \ j_b} &= \sqrt{\frac{j_a}{j_a+j_b}} \end{aligned}}$$

we can continue by applying $J^-_{ab} = J^-_a + J^-_b$ to both sides of this equation

this generates all states of the form

$$|j_a+j_b, \mu\rangle \qquad -(j_a+j_b) \le \mu \le (j_a+j_b)$$

* there are $2(j_a+j_b)+1$ such states
* for these states all of the Clebsch Gordan coefficients are positive

since the is only one state with $m_{ab} = j_a + j_b$ it is the state $|j_a j_a j_b j_b\rangle$

there are 2 states with $m_{ab} = j_a + j_b - 1$
there are

$$|j_a\, j_a\, j_b\, j_b - 1\rangle, \quad |j_a\, j_a - 1, j_b\, j_b\rangle$$

one of the states is $|j_a + j_b\;\; j_a + j_b - 1\rangle$

consider the orthogonal complement

$$|x\rangle = \sqrt{\frac{j_a}{j_a + j_b}}\, |j_a\, j_a\, j_b\, j_b - 1\rangle - \sqrt{\frac{j_b}{j_a + j_b}}\, |j_a\, j_a - 1\, j_b\, j_b\rangle$$

$$* \quad (J_a^z + J_b^z)\,|x\rangle = (j_a + j_b - 1)\,|x\rangle$$

$$J_{ab}^+ |x\rangle = \sqrt{\frac{j_a}{j_a + j_b}}\, |j_a\, j_a\, j_b\, j_b\rangle \times \sqrt{2 j_b}$$

$$- \sqrt{\frac{j_b}{j_a + j_b}}\, |j_a\, j_a\, j_b\, j_b\rangle \sqrt{2 j_a} = 0$$

$$J_{ab}^2\, |x\rangle = \left(J_- J_+ + (J_{ab\,z} + 1)(J_{ab\,z})\right) |x\rangle$$

$$= (j_a + j_b - 1)(j_a + j_b)\, |x\rangle$$

these equations imply that $|x\rangle$ is a eigenstate of $\bar{J}_{ab}^2$ and J_{ab}^z with eigenvalues

$$m_{ab} = j_a + j_b - 1 \qquad j_{ab} = j_a + j_b - 1$$

It is also a highest weight state. In this case while this state is normalized we need to assign a phase.

* we can choose the coefficients to be real by taking the normalization $= \pm 1$.

To choose the sign if $j_a \geq j_b$ choose the coefficient of $|j_a j_a, j_b j_b-1\rangle$ to be positive. This is just a convention - it is fairly standard (but not required in principle - called the Condon-Shortly convention)

Using the lowering operator we can then calculate all $2(j_a+j_b-1)+1$ states with $J = j_a+j_b-1$

Next we note there are 3 states with $j = j_a + j_b - 2$

$$|j_a\, j_a\, j_b\, j_b - 2\rangle \quad |j_a\, j_a - 1\, j_b\, j_b - 1\rangle \quad |j_a\, j_a - 2\, j_b\, j_b\rangle$$

There are 2 states with $j > j_a + j_b - 2$. It is possible to show that the orthogonal complement is a highest weight state of J^2_{ab} J^z_{ab} with $j = j_a + j_b - 2$.

* the normalization can again be fixed by the Condon Shortly convention

* Lowering operators can be used to construct $|j_a + j_b - 2, \mu\rangle$ for $-(j_a + j_b - 2) \leq \mu \leq (j_a + j_b - 2)$

This process can be continued until

$$j_{ab} = |j_a - j_b| = j_a - j_b \quad (\text{for } j_a \geq j_b)$$

This gives

$$N = \sum_{j=|j_a-j_b|}^{|j_a+j_b|} (2j+1) \overset{?}{=} (2j_a+1)(2j_b+1)$$

Homework

simultaneous eigenstates of J_{ab}^2 and J_{ab}^z

This generates all Clebsch-Gordon coefficients

Consider

$$\langle j_a \mu_a\, j_b \mu_b | e^{-i\bar{\theta}\cdot(J_a+J_b)} | j_a \nu_a\, j_b \nu_b \rangle =$$

$$\langle j_a \mu_a | e^{-i\bar{\theta}\cdot \bar{J}_a} | j_a \nu_a \rangle \langle j_b \mu_b | e^{-i\theta\cdot J_b} | j_b \nu_b \rangle =$$

$$D^{j_a}_{\mu_a \nu_a}(R)\, D^{j_b}_{\mu_b \nu_b}(R) =$$

$$\sum_{j_{ab},\mu_{ab}} \langle j_a \mu_a\, j_b \mu_b | j_{ab}\, \mu_{ab} \rangle \langle j_{ab}\, \mu_{ab} | e^{-i J_{ab}\cdot\bar{\theta}} | j_{ab} \nu_{ab} \rangle$$

$$\langle j_{ab}\, \nu_{ab} | j_a \nu_a\, j_b \nu_b \rangle =$$

$$= \sum_{j=|j_a-j_b|}^{j_a+j_b} \sum_{\mu,\nu=-j}^{j} C^{j_{ab}\ j_a\ j_b}_{\mu_{ab}\ \mu_a\ \mu_b} D^{j_{ab}}_{\mu_{ab}\nu_{ab}}(R)\, C^{j_{ab}\ j_a\ j_b}_{\nu_{ab}\ \nu_a\ \nu_b}$$

$$= D^{j_a}_{\mu_a\nu_a}(R)\, D^{j_b}_{\mu_b\nu_b}(R)$$

Note: each of $|j_a\mu_a\rangle$ and $|j_b\mu_b\rangle$ transform among themselves under rotations.

The coefficients $C^{j_{ab}\ j_a\ j_b}_{\mu_{ab}\ \mu_a\ \mu_b}$ block diagonalized this matrix into parts that transform irreducibly

$$D^{j_a}(R) \otimes D^{j_b}(R) = \bigoplus_{j=|j_a-j_b|}^{j_a+j_b} D^{j}(R)$$

$$C\left(D^{j_a}(R)\otimes D^{j_b}(R)\right) C^{+} = \begin{pmatrix} M_{j_a+j_b} & 0 & 0 & \cdots \\ 0 & M_{j_a+j_b-1} & \cdot & \cdots \\ 0 & 0 & & \\ 0 & 0 & & M_{|j_a-j_b|} \end{pmatrix}$$

The Clebsch Gordan coefficients block diagonalizes products of rotation matrices into invariant irreducible subspaces

Integrations

$$\int D^{j_a}_{\mu_a \nu_a}(R)\, D^{j_b}_{\mu_b \nu_b}(R)\, dR =$$

$$\sum_{j=|j_a-j_b|}^{j_a+j_b} \sum_{\mu=-j}^{j} \sum_{\nu=-j}^{j} C^{j\ j_a\ j_b}_{\mu\ \mu_a\ \mu_b} \int D^{j}_{\mu\nu}(R)\, dR\ C^{j\ j_a\ j_b}_{\nu\ \nu_a\ \nu_b}$$

$$= C^{0\ j_a\ j_b}_{0\ \mu_a\ \mu_b}\ C^{0\ j_a\ j_b}_{0\ \nu_a\ \nu_b}$$

(This vanishes unless $j_a = j_b$; $\mu_a = -\mu_b$, $\nu_a = -\nu_b$

This can be generalized to products of any number of $D^{j}_{\mu\nu}(R)$

It is also possible to perform integrals involving Wigner matrices and their complex conjugates

$$\int D^{j}_{\mu\nu}(R)\, D^{j'\,*}_{\mu'\nu'}(R)\, dR = \frac{\delta_{jj'}\,\delta_{\mu\mu'}\,\delta_{\nu\nu'}}{2j+1}$$

To show this note

(1) $\sigma_2 \bar{\sigma}^* \sigma_2 = -\bar{\sigma}$

$$R^* = \left(e^{-i\frac{\bar{\sigma}}{2}\cdot\bar{\theta}}\right)^* = \left(e^{i\frac{\sigma^*}{2}\theta}\right)$$

$$= \left(e^{i(-\sigma_2\bar{\sigma}\sigma_2)\cdot\theta/2}\right) =$$

$$= \sigma_2 \, e^{-i\bar{\sigma}\cdot\bar{\theta}/2} \, \sigma_2$$

$$= \sigma_2 R \sigma_2$$

(2) $R_y(\pi) = \cos(\frac{\pi}{2}) I - i\sin(\frac{\pi}{2})\cdot\sigma_2 = -i\sigma_2$

$$R_y^{+}(\pi) = i\sigma_2 = R_y^{-1}(\pi)$$

$$R^* = R_y(\pi) R R_y(-\pi)$$

(3) since $D^j_{\mu\nu}(R)^* = D^j_{\mu\nu}(R^*)$ (D has real coefficients)

$$D^j_{\mu\nu}(R)^* = D^j_{\mu\nu}(R^*) = D^j_{\mu\nu}(R_y(\pi) R R_y(-\pi)) =$$

$$\sum D^j_{\mu\alpha}(R_y(\pi)) \, D^j_{\alpha\beta}(R) \, D^j_{\beta\nu}(R_y(-\pi))$$

$$\int D^{j'}_{\mu'\nu'}(R)\, D^{j}_{\mu\nu}(R)^{*}\, dR =$$

$$\sum_{\alpha\beta} \int D^{j'}_{\mu'\nu'}(R)\, D^{j}_{\mu\alpha}(R_y(\pi))\, D^{j}_{\alpha\beta}(R)\, D^{j}_{\beta\nu}(R_y(\pi)^{-1})\, dR$$

$$\sum_{\alpha\beta} \int \sum C^{\tilde{j} j' j}_{\tilde{\mu} \mu' \alpha}\, C^{\tilde{j} j' j}_{\tilde{\nu} \nu' \beta}\, D^{\tilde{j}}_{\tilde{\mu}\tilde{\nu}}(R)\, D^{j}_{\mu\alpha}(R_y(\pi))\, D^{j}_{\beta\nu}(R_y(\pi)^{-1})$$

doing the integral sets $\tilde{j} = \tilde{\mu} = \tilde{\nu} = 0$

$$\sum_{\alpha\beta} C^{0 j' j}_{0 \mu' \alpha}\, C^{0 j' j}_{0 \nu' \beta}\, D^{j}_{\mu\alpha}(R_y(\pi))\, D^{j}_{\beta\nu}(R_y(\pi)^{-1})$$

For $C^{0 j' j}_{0 \mu' \alpha} \neq 0 \Rightarrow j' = j \quad \alpha = -\mu' \quad (\beta = -\nu')$

$$C^{0 j j}_{0 \mu' -\mu'}\, C^{0 j j}_{0 \nu' -\nu'}\, D^{j}_{\mu -\mu'}(R_y(\pi))\, D^{j}_{-\nu'\nu}(R_\pi(\pi)^{-1})\, \delta_{jj'}$$

at this point we need to consider the value of these quantities

$$R_y(\pi) = e^{-i\frac{\pi}{2}\sigma_y} = \cos\left(\tfrac{\pi}{2}\right) - i\sigma_y \sin\left(\tfrac{\pi}{2}\right) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$R_{++} = R_{--} = 0 \qquad R_{-+} = -R_{+-} = 1$$

$$R_y(\pi)^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$R_{++} = R_{--} = 0 \qquad R_{+-} = -R_{-+} = -1$$

inserting these in the expression for $D^{j}_{\mu\nu}(R)$ and using the value of $C^{0jj}_{0\mu-\mu}$ lead to

$$C^{0 j j}_{0 -\mu' \mu'}\, C^{0 j j}_{0 \nu' -\nu'}\, D^{j}_{\mu -\mu'}(R_y(\pi))\, D^{j}_{-\nu'\nu}(R_\pi(\pi)^{-1})\, \delta_{jj'}$$

$$= \frac{\delta_{jj'}\,\delta_{\mu\mu'}\,\delta_{\nu\nu'}}{2j+1}$$

This gives

$$\int D^{j'}_{\mu'\nu'}(R)\, D^{*\,j}_{\mu\nu}(R)\, dR = \frac{\delta_{jj'}\delta_{\mu\mu'}\delta_{\nu\nu'}}{2j+1}$$

$$\int D^{j}_{\mu\nu}(R)\, dR = \delta_{j0}\delta_{\mu 0}\delta_{\nu 0}$$

using these relations and the Clebsch equation we can compute integrals over arbitrary products of $D(R)$ and $D^*(R)$ - with the Clebsch-Gordan coefficients can reduced to one of the above integrals

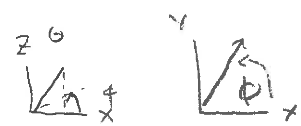
Note

$$Y_\ell^m(\theta\phi) = \langle \hat{r} | \ell m \rangle$$

$\hat{r}$ = unit vector with polar angles $\theta\,\phi$.

Active transformation

$$\hat{r} = R_z(\phi) R_y(\theta) \hat{z}$$



$$\langle \mathcal{U}(R)\hat{z} | \ell m \rangle =$$

$$\langle \hat{z} | \mathcal{U}^\dagger(R) | \ell m \rangle =$$

$$\sum \langle z | \ell m' \rangle \langle \ell m' | \mathcal{U}^\dagger(R) | \ell m \rangle$$

$$\sum \langle z | \ell m' \rangle \langle \ell m | \mathcal{U}(R) | \ell m' \rangle^*$$

where

$$\langle z | \ell m \rangle = Y_\ell^m(0,0) = \delta_{m0}\sqrt{\frac{2\ell+1}{4\pi}}$$

$$Y_\ell^m(\theta\phi) = \sqrt{\frac{2\ell+1}{4\pi}}\; D^{\ell\,*}_{m0}(R_z(\phi)R_y(\theta))$$

note

$$D^{\ell}_{m0}(R_z(\phi) R_y(\theta)) = D^{\ell}_{m0}(R_z(\phi) R_y(\theta) R_z(\psi))$$

$$= e^{-im\phi} D^{\ell}_{m0}(R_y(\theta)) \underbrace{e^{-i0\cdot\psi}}_{1}$$

$$= D^{\ell}_{m0}(R)$$

The means that we can use this to compute integrals involving D(R) and spherical harmonics

$$\int Y^{m}_{\ell}(\theta\phi) Y^{m'}_{\ell'}(\theta\phi) \sin\theta d\theta d\phi =$$

$$8\pi \cdot \int Y^{m}_{\ell}(\theta\phi) Y^{m'}_{\ell'}(\theta\phi)\, dR =$$

$$8\pi \times \sqrt{\frac{2\ell+1}{4\pi}} \sqrt{\frac{2\ell'+1}{4\pi}}\, D^{\ell *}_{m0}(R)\, D^{\ell' *}_{m'0}(R)\, dR$$

$$2\sqrt{(2\ell+1)(2\ell'+1)}\, C^{\tilde{\ell}\ell\ell'}_{\tilde{m}mm'}\, C^{\tilde{\ell}\ell\ell'}_{\tilde{n}00}\, D^{\tilde{\ell}}_{\tilde{m}\tilde{n}}(R)\, dR$$

$$2\sqrt{(2\ell+1)(2\ell'+1)}\, C^{0\ell\ell'}_{0mm'}\, C^{0\ell\ell'}_{000}$$

This requires $\ell = \ell'$ $m' = -m$

This can be generalized to more complicated integrals

This can be abstracted to any quantity $V^{i_1 \cdots i_n}$ that transforms like

$$V^{i_1 \quad i_n} \rightarrow V'^{i_1 \quad i_n} =$$

$$\sum V^{j_1 \quad j_n} R_{j_1 i_1} \quad R_{j_n i_n}$$

is called a rank n spherical tensor operator

consider $x^i x^j$ the dot product $\delta_{ij} x^i x^j$ and cross product $\varepsilon_{ijk} x^j x^k$ transform like scalars and vector under rotations

the remaining 5 independent component transform like a traceless symmetric tensor

the scalar, vector and tensor transform independently under rotations

note

$$Y^1_0 = \sqrt{\frac{3}{4\pi}} \cos\theta \qquad Y^1_1 = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\varphi} \qquad Y^{-1}_1 = \sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\varphi}$$

$$= \sqrt{\frac{3}{4\pi}} \frac{z}{r} \qquad = -\sqrt{\frac{3}{4\pi}} \frac{x+iy}{\sqrt{2}} \qquad = \sqrt{\frac{3}{4\pi}} \frac{x-iy}{\sqrt{2}}$$

For the vector $\bar{x} = x, y, z$ define the spherical components of $\bar{x}$ be

$$x^0_1 = z \qquad x^1_1 = -\frac{x-iy}{\sqrt{2}} \qquad x^{-1}_1 = \frac{x-iy}{\sqrt{2}}$$

Spherical Tensors

Consider a charge density $\rho(\bar{r})$ and the associated potential

$$V(\bar{r}) = \int \frac{\rho(r')}{|\bar{r}-\bar{r}'|}\, dr' = \frac{1}{r}\int \frac{\rho(r')}{\sqrt{1-2\hat{r}\cdot\frac{\bar{r}'}{r} + \left(\frac{\bar{r}'}{r}\cdot\frac{\bar{r}'}{r}\right)}}$$

for a finite charge density with $|r| >>$ than the size of the charge distribution $\left|\frac{\bar{r}'}{r}\right| << 1$ $\quad \bar{u} = \frac{\bar{r}'}{r}$

$$\frac{1}{\sqrt{}} \to 1 + \frac{\partial_i}{\partial u}\left(\frac{1}{\sqrt{}}\right)(0)\,\frac{x_i'}{r} + \frac{1}{2!}\frac{\partial^2}{\partial u_i \partial u_j}\left(\frac{1}{\sqrt{}}\right)0\,\frac{x_i' x_j'}{r^2}$$

$$+ \frac{1}{3!}\frac{\partial^3}{\partial u_i \partial u_j \partial u_k}\left(\frac{1}{\sqrt{}}\right)\frac{x_i' x_j' x_k'}{r^3} + \cdots$$

The important thing is that this multipole expansion involves coefficients multiplied by quantities that look like products of vectors

If we consider $V(R\bar{r})$ each one of the components rotates

$$x_i x_j x_k \to x_i' x_j' x_k' = \sum R^T_{ii'} R^T_{jj'} R^T_{kk'}\, x_{i'} x_{j'} x_{k'}$$
$$= \sum x_{i'} x_{j'} x_{k'} R_{i'i} R_{j'j} R_{k'k}$$

The terms in the multipole expansion are tensors

$$\left(V(Rr) = \int \frac{\rho(r')}{|Rr - r'|}\, dr' = \int \frac{\rho(r')}{|r - R^{-1}r'|} \right)$$

these transform like

$$\mathcal{U}(R)\, X_1^m\, \mathcal{U}^\dagger(R) = \sum X_1^{m'}\, D^1_{m'm}(R)$$

clearly we can extract the cartesian components from the spherical components

We generalize this to define a spherical tensor operator as an operator that transforms like

$$\boxed{\mathcal{U}(R)\, X_j^\mu\, \mathcal{U}^\dagger(R) = \sum_\nu X_j^\nu\, D^j_{\nu\mu}(R)}$$

The Wigner Eckart Theorem

consider matrix elements of a spherical tensor operator in an angular momentum basis

$$\langle j_a \nu_a | V^{j_b}_{\nu_b} | j_c \nu_c \rangle$$

Claim: this is proportional to

$$C^{j_a\, j_b\, j_c}_{\nu_a\, \nu_b\, \nu_c}$$

proof

$$\langle j_a \nu_a | V_{j_b}^{\nu_b} | j_c \nu_c \rangle =$$

$$\int \langle j_a \nu_a | \mathcal{U}^\dagger(R) \mathcal{U}(R) V_{j_b}^{\nu_b} \mathcal{U}^\dagger(R) \mathcal{U}(R) | j_c \nu_c \rangle \, dR$$

$$\sum \int D_{\mu_a \nu_a}^{j_a}(R)^* \langle j_a \mu_a | V_{j_b}^{\mu_b} | j_c \mu_c \rangle D_{\mu_b \nu_b}^{j_b}(R) D_{\mu_c \nu_c}^{j_c}(R) \, dR$$

$$\sum \int D_{\mu_c \nu_c}^{j_a}{}^*(R) \langle j_a \mu_a | V_{j_b}^{\mu_b} | j_c \mu_c \rangle \times$$

$$C_{\mu \mu_b \mu_c}^{j\, j_b\, j_c} C_{\nu \nu_b \nu_c}^{j\, j_b\, j_c} D_{\mu c}^{j}(R) \, dR$$

$$\frac{1}{2j_c+1} C_{\nu_a \nu_b \nu_c}^{j_a\, j_b\, j_c} \Big[\underbrace{\sum C_{\mu_a \mu_c \mu}^{j_a\, j_b\, c_c} \langle j_a \mu_a | V_{j_b}^{\mu_b} | j_c \mu_c \rangle}_{\text{all of the } \mu\text{'s are summed here so this is a constant that depends on } j_a\, j_b\, j_c} \Big]$$

$$= \frac{C^{j_a\, j_b\, j_c}}{\sqrt{2j_c+1}} \times \langle j_a \| V_{j_b} \| j_c \rangle$$

↑ reduced matrix element

where

$$\langle j_a \| V_{j_b} \| j_c \rangle = \frac{1}{\sqrt{2j_c+1}} \sum_{\mu_a \mu_b \mu} C_{\mu_a \mu_b \mu_c}^{j_a\, j_b\, j_c} \langle j_a \mu_a | V_{j_b}^{\mu_b} | j_c \mu_c \rangle$$

since $C^{j_a j_b j_c}_{\nu_a \nu_b \nu_c} = 0$ unless

$$\left.\begin{array}{l} |j_b - j_c| \le j_a \le |j_b + j_c| \\ \nu_a = \nu_b + \nu_c \end{array}\right\} \text{selection rules}$$

$$\frac{\langle j_a \nu_a | V^{\nu_b}_{j_b} | j_b \nu_c \rangle}{\langle j_a \nu_a' | V^{\nu_b'}_{j_b} | j_b \nu_b' \rangle} = \frac{C^{j_a j_b j_c}_{\nu_a \nu_b \nu_c}}{C^{j_a j_b j_c}_{\nu_a' \nu_b' \nu_c'}}$$