Lecture 4

Lost time

$$\mathcal{U}(R) = e^{-i\vec{J}\cdot\vec{\theta}}$$

$$[J_i, J_j] = i \sum \varepsilon_{ijk} J_k$$

$$[\vec{J}^2, J_z] = 0 \quad \Rightarrow |j\,\mu\rangle$$

$$J^2 |j\mu\rangle = j(j+1)|j\mu\rangle \qquad j = \frac{n}{2} \quad n = 0,1,2\ldots$$

$$J_z |j\mu\rangle = \mu |j\mu\rangle \qquad -j \le \mu \le j$$

$$n_\pm = j \pm \mu \qquad |n_+ n_-\rangle = |j\mu\rangle$$

$$j = \tfrac{1}{2}(n_+ + n_-) \qquad \mu = \tfrac{1}{2}(n_+ - n_-)$$

$$a_+ |n_+ n_-\rangle = |n_+ + 1\ n_-\rangle \sqrt{n_+}$$

$$a_- |n_+ n_-\rangle = |n_+\ n_- - 1\rangle \sqrt{n_-}$$

$$a_+^\dagger |n_+ n_-\rangle = |n_+ + 1\ n_-\rangle \sqrt{n_+ + 1}$$

$$a_-^\dagger |n_+ n_-\rangle = |n_+\ n_- + 1\rangle \sqrt{n_- + 1}$$

$$J_z = \tfrac{1}{2}(a_+^\dagger\ a_-^\dagger)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}a_+\\ a_-\end{pmatrix}$$

$$J_\pm = a_\pm^\dagger a_\mp \Rightarrow \vec{J} = \tfrac{1}{2}(a_+\ a_-)\vec{\sigma}\begin{pmatrix}a_+\\ a_-\end{pmatrix}$$

$$|n_+ n_-\rangle = \frac{(a_+^\dagger)^{n_+}}{\sqrt{n_+!}}\frac{(a_-^\dagger)^{n_-}}{\sqrt{n_-!}}$$

$$D^j_{\mu\nu}(R) = \langle j\mu| e^{-i\vec{J}\cdot\vec{\theta}} |j\nu\rangle = \langle n_+' n_-'| e^{-i\frac{\vec{\theta}}{2}\cdot a^\dagger\vec{\sigma} a} |n_+ n_-\rangle$$

$$= \frac{\langle 00|(a_+)^{n_+'}(a_-)^{n_-'}}{\sqrt{n_+'!\,n_-'!\,n_+!\,n_-!}}\left(a_+^\dagger R_{++} + a_-^\dagger R_{-+}\right)^{n_+}\left(a_+^\dagger R_{+-} + a_-^\dagger R_{--}\right)^{n_-}|00\rangle$$

where $R_{ij} = \left(I\cos\left(\frac{\theta}{2}\right) - i\,\hat{\theta}\cdot\vec{\sigma}\sin\left(\frac{\theta}{2}\right)\right)_{ij}$

next we use the binomial theorem to calculate

$$(a_+^\dagger R_{++} + a_-^\dagger R_{-+})^{n_+}$$
$$(a_+^\dagger R_{+-} + a_-^\dagger R_{--})^{n_-}$$

where

$$(a+b)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^k b^{n-k}$$

applying this to the expressions above gives

$$(a_+^\dagger R_{++}\; a_-^\dagger R_{-+})^{n_+} =$$
$$\sum_{k=0}^{n_+} \frac{(n_+)!}{k!(n_+-k)!} (a_+^\dagger R_{++})^k (a_-^\dagger R_{-+})^{n_+-k}$$

with a similar expression for

$$(a_+^\dagger R_{+-} + a_-^\dagger R_{--})^{n_-}$$

using these in the expression for $D^j_{uv}(R)$ gives

$$\sum_{k,m} \frac{n_+!\, n_-!}{\sqrt{n_+'!\, n_-'!\, n_+!\, n_-!}} \frac{1}{k!\,(n_+-k)!\,m!\,(n_--m)!} R_{++}^{k} R_{-+}^{n_+-k} R_{+-}^{n_--m} R_{--}^{m}$$

$$\underbrace{\langle 00| (a_+)^{n_+'} (a_-)^{n_-'} (a_+^\dagger)^{k+n_--m} (a_-^\dagger)^{m+n_+-k} |00\rangle}_{n_+'!\, n_-'!}$$

where $n_+' = k+n_--m \quad n_-' = m+n_+-k$

we can use these relations to eliminate m

$$m = k + n_- - n'_+$$

we also have

$$n_\pm = j \pm v \qquad n'_\pm = j \pm u$$

$$\begin{cases} n_+ - k = j + v - k \\ n_- - m = n_- - (k + n_- - n'_+) = n'_+ - k \\ m = k + n_- - n'_+ \end{cases}$$

$$\boxed{\begin{aligned} D^j_{uv}(R) &= \sum_{k=0}^{n_+} \frac{\sqrt{n'_+!\, n'_-!\, n_+!\, n_-!}}{k!\,(j+v-k)!\,(n'_+-k)!\,(k+n_--n'_+)!}\, R_{++}^{k}\, R_{-+}^{n_+-k}\, R_{+-}^{n'_+-k}\, R_{++}^{k+n_--n'_+} \\ &= \sum_{k=0}^{j+v} \frac{\sqrt{(j+u)!\,(j-u)!\,(j+v)!\,(j-v)!}}{k!\,(j+v-k)!\,(j+u-k)!\,(k-v-u)!}\, R_{++}^{k}\, R_{-+}^{j+v-k}\, R_{+-}^{j+u-k}\, R_{--}^{k-v-u} \end{aligned}}$$

exercise: show

$$D^{\frac{1}{2}}_{uv}(R) = R_{uv}.$$

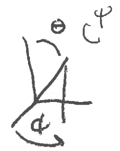
where R is an $SU(2)$ matrix

observations

(1) $D^j_{uv}(R)$ is a homogeneous polynomial of degree $2j$ in the components of R_{ij}

(2) The polynomial has real coefficients

(3) To cover all possible values of R



ϕ, ψ must range between $0, 4\pi$

Adding angular momenta

assume

$$[J_a^i\ J_a^j] = i\sum \varepsilon_{ijk} J_a^k$$
$$[J_b^i\ J_b^j] = i\sum \varepsilon_{ijk} J_b^k$$
$$[\bar{J}_a, \bar{J}_b] = 0$$

Homework: show

$$\bar{J}_{ab} = \bar{J}_a + \bar{J}_b$$
$$[J^i_{ab}, J^j_{ab}] = i\sum_k J^k_{ab}$$

This means $\bar{J}_{ab}$ is also an angular momentum operator

consider:

$$J^z_{ab}\,|j_a j_a\ j_b j_b\rangle = (j_a + j_b)\,|j_a j_a\ j_b j_b\rangle$$

$$J_{ab+}\,|j_a j_a\ j_b j_b\rangle = 0. \qquad \text{(highest weight)}$$

$$J_{-ab} J_{ab+} = J^2_{ab} - (J^z_{ab})^2 - J^z_{ab}$$

$$J^2_{ab} = J_{-ab} J_{+ab} + (J^z_{ab})^2 + J^z_{ab}$$

This means

$$J^2_{ab}\,|j_a j_a\, j_b j_b\rangle =$$

$$(\underbrace{J_{-ab}J_{+ab}}_{0} + \underbrace{(J^z_{ab})^2}_{(j_a+j_b)^2} + \underbrace{J^z_{ab}}_{(j_a+j_b)})\,|j_a j_a\, j_b j_b\rangle$$

$$(j_a+j_b)(j_a+j_b+1)\,|j_a j_a\, j_b j_b\rangle$$

we also have

$$\langle j_a j_a\, j_b j_b | j_a j_a\, j_b j_b\rangle = 1$$

$$\therefore\quad |j_a+j_b,\, j_a+j_b\rangle = |j_a j_a\, j_b j_b\rangle$$

(simultaneous eigenstate of J^2_{ab} and J^z_{ab})

since $|j_{ab}\, m_{ab}\rangle$; $|j_a m_a\, j_b m_b\rangle$ are 2 orthonormal bases for the $(2j_a+1)(2j_b+1)$ dimension vector space spanned by $|j_a m_a\, j_b m_b\rangle$ we have

$$|j_{ab}\, m_{ab}\rangle = \sum |j_a m_a\, j_b m_b\rangle\langle j_a m_a\, j_b m_b | j_{ab}\, m_{ab}\rangle$$

$$|j_a m_a\, j_b m_b\rangle = \sum |j_{ab}\, m_{ab}\rangle \underbrace{\langle j_{ab}\, m_{ab} | j_a m_a\, j_b m_b\rangle}_{\langle j_a m_a\, j_b m_b | j_{ab}\, m_{ab}\rangle^*}$$

* we will choose phases so $\langle j_a m_a\, j_b m_b | j_{ab}\, m_{ab}\rangle$ is real!!

* There are no other state in this space with $m_{ab} = j_a + j_b$

* Once we know $|j_a+j_b \; j_a+j_b\rangle$ it is possible to use lowering operators to construct the $2(j_a+j_b)+1$ states of the form $|j_a+j_b, m\rangle \quad -(j_a+j_b) \le m \le (j_a+j_b)$

$$J_{ab}^{-} |j_a+j_b \; j_a+j_b\rangle = (J_a^{-} + J_b^{-}) |j_a j_a \; j_b j_b\rangle$$

using

$$J_{\pm} |j m\rangle = |j \, m\pm 1\rangle \sqrt{(j \mp m)(j \pm m + 1)}$$

$$\sqrt{2(j_a+j_b)(1)} \; |j_a+j_b, j_a+j_b-1\rangle =$$

$$\sqrt{2j_a} \; |j_a \, j_a-1 \; j_b \, j_b\rangle + \sqrt{2j_b} \; |j_a \, j_a \; j_b \, j_b-1\rangle$$

This gives

$$|j_a+j_b, j_a+j_b-1\rangle = \underbrace{\sqrt{\frac{j_a}{j_a+j_b}} \; |j_a \, j_a-1 \; j_b \, j_b\rangle + \sqrt{\frac{j_b}{j_a+j_b}}}_{\text{These are the Clebsch Gordan coefficients}} \; |j_a \, j_a \; j_b \, j_b-1\rangle$$

$\langle j_a\, j_a-1\; j_b\, j_b \mid j_a+j_b, j_a+j_b-1\rangle = \sqrt{\frac{j_a}{j_a+j_b}}$

$\langle j_a\, j_a\; j_b\, j_b-1 \mid j_a+j_b, j_a+j_b-1\rangle = \sqrt{\frac{j_b}{j_a+j_b}}$

This gives one state with $M = j_a+j_b-1$
however there are 2 independent states with $M = j_a+j_b-1$

$|j_a\, j_a-1, j_b\, j_b\rangle$

and

$|j_a\, j_a\; j_b\, j_b-1\rangle$

The next independent state $\perp$ to $|j_a+j_b, j_a+j_b-1\rangle$

$\alpha|a\rangle + \beta|b\rangle = |A\rangle$

$\beta|a\rangle - \alpha|b\rangle = |B\rangle$

$\langle A|B\rangle = (\alpha\beta - \beta\alpha) = 0$

we defin

$| ?, j_a+j_b-1\rangle = \sqrt{\frac{j_b}{j_a+j_b}}\,|j_a\, j_a\; j_b\, j_b-1\rangle - \sqrt{\frac{j_a}{j_a+j_b}}\,|j_a\, j_a-1\; j_b\, j_b\rangle$

* This is the only other state of $J_{z\,ab}$ with eigenvalue j_a+j_b-1

show

(1) $J_{+ab} \, | ? \; j_a + j_b - 1 \rangle = 0$

(2) $J^2_{ab} \, | ? \; j_a + j_b - 1 \rangle = (j_a + j_b - 1)(j_a + j_b) \, | ? \; j_a + j_b - 1 \rangle$

This means that

$$| ? \; j_a + j_b - 1 \rangle = c \, | j_a + j_b - 1 , \; j_a + j_b - 1 \rangle$$

(1) this state is normalized $|c|^2 = 1$

(2) the coefficients are real $c = \pm 1$

(3) how to choose the overall sign is a convention

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* Condon Shortly convention

$j_a \geq j_b$

coefficient of $| j_a j_a \, j_b \, j_b - a \rangle$ is positive

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This process can be repeated until

$$J_{ab} = |j_a - j_b|$$

This generates all Clebsch Gordan coefficients. By construction they are real

general remarks

Homework

show $\quad 2(j_a+j_b)+1 = (2j_a+1)(2j_b+1)$

(This shows that both bases have the same number of independent vectors)

Comments on Clebsch Gordan coefficients

* The are real
* They are coefficients of a unitary change of basis
* The notation $C^{j_{ab}\ j_a\ j_b}_{\mu_{ab}\ \mu_a\ \mu_b} \Longrightarrow$

(1) $j_a \geq j_b$

(2) $C^{j_{ab}\ j_a\ j_b}_{\mu_{ab}\ j_a\ \mu_{ab}-j_a} \geq 0 \quad$ (Condon Shortley)

(3) It is easy to make sign mistakes using $\langle j_{ab}\,\mu_{ab} \mid j_a\, j_a\, j_b\, \mu_{ab}-j_a \rangle$

Group integration

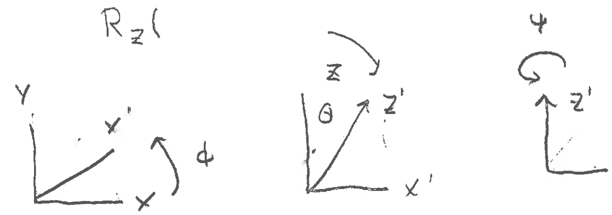
* any rotation R can be expressed as $\quad R = R R_0^{-1} R_0 = R' R_0 \quad$ where R_0 is a fixed rotation

* If we average over all rotations with equal weight

$$\int D^j_{\mu\nu}(R)\,dR = \int D^j_{\mu\nu}(R'R_0)\,dR = \int_{\mu\nu}^{j} D^j_{\mu\nu}(R'R_0)\,dR'$$

what do we mean by dR

$R = \qquad R_z($



$$R_z(\psi)\,R_y(\theta)\,R_z(\phi)$$

$\psi: 0 \to 4\pi \quad \theta: 0 \to \pi \quad \phi: 0 \to 4\pi$

$$= \int D^j_{\mu\alpha}(R')\,dR'\; D_{\alpha\nu}(R_0)$$

$$\therefore \quad \int D^j_{\mu\nu}(R)\,dR = \int D^j_{\mu\alpha}(R)\,dR\; D_{\alpha\nu}(R_0)$$

↑ fixed arbitrary R_0

$$M^J_{\mu\nu} = M^J_{\nu\alpha}\, D^J_{\alpha\nu}(R_0)$$

The left side of this expression is clearly independent of R_0. — this requires either

$$M^j_{\mu\nu} = 0 \quad \text{or} \quad j = \alpha = \nu = 0$$

This gives

$$\int D^j_{\mu\nu}(R)\,dR = C\,\delta_{j0}\,\delta_{\mu 0}\,\delta_{\nu 0}$$

The constant c is a normalization constant that depends on dR

For $\quad dR = \dfrac{\sin\theta \, d\theta \, d\phi \, d\psi}{4\pi \cdot 4\pi \cdot 2} \qquad \int dR = 1$

$$R = R_z(\phi) R_y(\theta) R_z(\psi)$$

dR is called Haar measure

$$\boxed{\int D^{j}_{\mu\nu}(R)\, dR = \delta_{j0}\delta_{\mu 0}\delta_{\nu 0}}$$

What this equation means

$$|\psi\rangle = \sum c_\mu |j\mu\rangle$$

$$U(R)|\psi\rangle = \sum \mathcal{U}(R)|j\mu\rangle c_\mu$$

$$= \sum |j\nu\rangle\langle j\nu|\mathcal{U}(R)|j\mu\rangle c_\mu$$

$$= \sum |j\nu\rangle D^{j}_{\mu\nu}(R) c_\mu$$

$$\int \mathcal{U}(R)|\psi\rangle\, dR = \sum_{\mu\nu} |j\nu\rangle \delta_{j0}\delta_{\mu 0}\delta_{\nu 0} c_\mu$$

vanishes unless $j = \mu = \nu = 0$, - this means that except of the identity and the whole $2j+1$ dimensions space spanned by $|j\mu\rangle$, the are no non trivial rotationally invariant subspaces

Tensor product

$$\mathcal{U}_{ab}(\bar{\Theta}) = e^{-i(\bar{J}_a + \bar{J}_b)\cdot\bar{\Theta}} =$$

$$e^{-i\bar{J}_a\cdot\bar{\Theta}}\, e^{-i\bar{J}_b\cdot\bar{\Theta}} = \mathcal{U}_a(\Theta)\,\mathcal{U}_b(\Theta) \qquad \text{since } [\bar{J}_a, \bar{J}_b] = 0$$

$$\langle j_{ab}\, \mu_{ab} | \mathcal{U}_{ab}(\Theta) | j_{ab}\, \nu_{ab} \rangle =$$

$$\langle j_{ab}\, \mu_{ab} | j_a \mu_a\, j_b \mu_b \rangle\, D^{j_a}_{\mu_a \nu_a}(R)\, D^{j_b}_{\mu_b \nu_b}(R) \times$$

$$\langle j_a \nu_a\, j_b \nu_b | j_{ab}\, \nu_{ab} \rangle$$

$$\boxed{D^{j_{ab}}_{\mu_{ab}\nu_{ab}}(R) = \sum_{\substack{\mu_a \nu_a \\ \mu_b \nu_b}} C^{j_{ab}\, j_a\, j_b}_{\mu_{ab}\, \mu_a\, \mu_b}\, D^{j_a}_{\mu_a \nu_a}(R)\, D^{j_b}_{\mu_b \nu_b}(R)\, C^{j_{ab}\, j_a\, j_b}_{\nu_{ab}\, \nu_a\, \nu_b}}$$

Since the clebsch gordon are coefficients of unitary transformation

$$\boxed{D^{j_a}_{\mu_a \nu_a}(R)\, D^{j_b}_{\mu_b \nu_b}(R) = \sum_{j_{ab}, \mu_{ab} \nu_{ab}} C^{j_{ab}\, j_a\, j_b}_{\mu_{ab}\, \mu_a\, \mu_b}\, D^{j_{ab}}_{\mu_{ab} \nu_{ab}}(R)\, C^{j_{ab}\, j_a\, j_b}_{\nu_{ab}\, \nu_a\, \nu_b}}$$

This decomposes the space spanned by $|j_a \mu_a\rangle$ and $|j_b \mu_b\rangle$ into a orthogonal direct sum of invariant subspaces of dimension $2j+1$ $\quad |j_a - j_b| \le j \le |j_a + j_b|$

$D^j_{\mu\nu}(R)$ defines a $2j+1$ dimensional irreducible representation (no non trivial invariant subspaces) of $SU(2)$

Theorem

$$\int D^j_{\mu\nu}(R)\, D^{*j'}_{\mu'\nu'}(R)\, dR = \frac{\delta_{jj'}\,\delta_{\mu\mu'}\,\delta_{\nu\nu'}}{2j+1}$$

step 1

$$R = e^{-i\frac{\bar{\sigma}}{2}\cdot\bar{\theta}}$$

$$R^* = e^{i\frac{\sigma^*}{2}\cdot\theta} = e^{-i\frac{\sigma_2\sigma\sigma_2}{2}\cdot\theta} = \sigma_2\, e^{-i\frac{\sigma\cdot\theta}{2}}\,\sigma_2$$

step 3

$$R_y(\pi) = e^{-i\frac{\sigma_2}{2}\pi} = \cos\left(\tfrac{\pi}{2}\right) - i\sigma_2 \sin\left(\tfrac{\pi}{2}\right) = -i\sigma_2$$

$$R^* = R_y(\pi)\, R\, R_y^{+}(\pi)$$

$$D^j_{\mu\nu}(R^*) = \left(D^j_{\mu\nu}(R)\right)^* \quad \text{since } D \text{ has real coefficient}$$

$$= \left(D(R_y(\pi))\, D(R)\, D^{+}(R_y(\pi))\right)$$

(homogeneous polynomial $\Rightarrow$ we can forget the i

step 4 use explicit expression for $D^j_{\mu\nu}(R)$

$$R_{++} = R_{--} = 0 \qquad R_{+-} = 1 \quad R_{-+} = -1$$

$R_{++} = 0 \Rightarrow k = 0$ (in $D^j_{\mu\nu}(R)$ on page L9-3

$R_{--} = 0 \Rightarrow \mu + \nu = 0$

$R_{+-}^{\,j+\mu}$

$R_{-+}^{\,j+\nu} = R_{-+}^{\,j-\mu}$

$$D^j_{\mu\nu}(R_y(\pi)) = \frac{(j+\mu)!\,(j+\nu)!}{(j+\mu)!\,(j+\nu)!}(-)^{j+\mu}(+)^{j-\mu} = (-)^{j+\mu}$$

$$D^j_{\mu\nu}(R_y^\dagger(\pi)) = \frac{(j+\mu)!\,(j+\nu)!}{(j+\mu)!\,(j+\nu)!}(+)^{j+\mu}(-)^{j-\mu} = (-)^{j-\mu}$$

This means

$$D^j_{\mu\nu}(R)^* = D^j_{\mu\alpha}(R_y(\pi))\, D^j_{\alpha\beta}(R)\, D^j_{\beta\nu}(R_y^\dagger(\pi))$$

$$= (-)^{j+\mu}\delta_{\mu,-\alpha}\, D^j_{\alpha\beta}(R)\,\delta_{\beta,-\nu}(-)^{j+\nu}$$

$$= (-)^{2j+\mu+\nu} D^j_{-\mu\,-\nu}(R)$$

next consider

$$\int D^{j_a}_{\mu_a\nu_a}(R)\, D^{j_b}_{\mu_b\nu_b}(R)\, dR =$$

$$(-)^{2j_b+\mu_b+\nu_b} C^{j_{ab}\ j_a\ j_b}_{\mu_{ab}\ \mu_a\ \mu_b} C^{j_{ab}\ j_a\ j_b}_{\nu_{ab}\ \nu_a\ \nu_b} \int D^{j_{ab}}_{\mu_{ab}\nu_{ab}}(R)\, dR$$

$$(-)^{2j_b+\mu_b+\nu_b} C^{0\ j_a\ j_b}_{0\ \mu_a\ -\mu_a} C^{0\ j_a\ j_b}_{0\ \nu_a\ -\nu_a} \Rightarrow j_a = j_b$$

$\mu_a = \mu_b$

$\nu_a = \nu_b$

since

$\nu_a - \nu_b = 0$

$\mu_a - \mu_b = 0$

$$(-)^{2j_a+\mu_a+\mu_b} \underbrace{C^{0\ j_a\ j_b}_{0\ \mu_a-\mu_b}}_{\frac{1}{\sqrt{2j_a+1}}} \underbrace{C^{0\ j_a\ j_b}_{0\ \nu_a-\nu_b}}_{\frac{1}{\sqrt{2j_a+1}}}$$

$$\underbrace{(-)^{2(j_a+\mu_a)}}_{1}$$

This gives

$$\int dR\; D^{j}_{\mu\nu}(R)\, D^{j'}_{\mu'\nu'}(R)^{*}\, dR = \frac{\delta_{jj'}\,\delta_{\mu\mu'}\,\delta_{\nu\nu'}}{2j+1}$$