Lecture 41 - Identical Particles

Occupation # representation

$\{|\phi_n\rangle\}_{n=1}^{\infty}$ single particle basis

$|0\rangle$ 0 particle state

$\langle 0|0\rangle = 1$ normalization

operators

a_n^+ adds one boson in state $|\phi_n\rangle$ to system

b_n^+ adds one fermion in state $|\phi_n\rangle$ to system

normalization of operator

$[a_n, a_m^+] = \delta_{mn}$

$\{b_n, b_m^+\} = \delta_{mn}$

properties of operator algebra

① $[a_n (a_n^\dagger)^m] = m(a_n^\dagger)^{m-1}$

proof by induction

$m=1 \quad [a_n a_n^\dagger] = 1$

$m = k$

$$[a_n, (a_n^\dagger)^{k+1}] =$$

$$a_n (a_n^\dagger)^{k+1} - (a_n^\dagger)^{k+1} a_n =$$

$$(\underbrace{a_n a_n^\dagger - a_n^\dagger a_n}_{1} + a_n^\dagger a_n)(a_n^\dagger)^k - a_n^\dagger (a_n^\dagger)^k a_n$$

$$(a_n^\dagger)^k + a_n^\dagger (\underbrace{a_n (a_n^\dagger)^k - (a_n^\dagger)^k a_n}_{\substack{\text{induction} \\ k(a_n^\dagger)^{k-1}}}) =$$

$$(1+k)(a_n^\dagger)^k$$

which shows that if this holds for k it holds for $k+1$

② $[a_n^\dagger a_n, (a_n^\dagger)^m] =$

$$a_n^\dagger a_n (a_n^\dagger)^m - (a_n^\dagger)^m a^\dagger a_n =$$

$$a_n^\dagger [a_n, (a_n^\dagger)^m] = a_n^\dagger m (a_n^\dagger)^{m-1} =$$

(by the previous result)

$$m (a_n^\dagger)^m \qquad \boxed{N_n = a_n^\dagger a_n}$$

③ $(b_n^\dagger)^m = 0$ unless $m = 0$ or 1

$$\{b_n, b_n^\dagger\} = 1$$

states in the occupation number representation

$$|m_1 m_2 \cdots \rangle$$

↑

m_j = # of identical particles in state $|\phi_j\rangle$

Note for Fermions the ordering of the states matter

Normalization choice

$$\langle m_1 m_2 \cdots | m_1' m_2' \cdots \rangle = \delta_{m_1 m_1'} \delta_{m_2 m_2'} \cdots$$

Relation to normalization on a_n a_n^+ b_n b_n^+

$$(a_n^+)^m |0\rangle = |0 \cdots \underset{\underset{n}{\downarrow}}{m} \cdots \rangle N(m)$$

$$N(m)^2 = \langle 0|(a_n)^m (a_n^+)^m |0\rangle =$$

$$\langle 0| a_n^{m-1} \, m (a_n^+)^{m-1} |0\rangle =$$

$$m(m-1) \langle 0| a_n^{m-2} (a_n^+)^{m-2} |0\rangle = \cdots$$

$$m! \langle 0|0\rangle$$

This means that $N(m) = \sqrt{m!}$

For fermions

$$b_n^{+}|0\rangle = |0 \cdots \underset{n}{1} \cdots\rangle N$$

In this case

$$\langle 0| b_n b_n^{+} |0\rangle = N^2 =$$

$$\langle 0| (b_n b_n^{+} + \underbrace{b_n^{+} b_n}_{0}) |0\rangle$$

$$\langle 0|0\rangle \cdot 1 = N^2 \quad \text{so} \quad N = 1$$

In both cases

$$\frac{(a_n^{+})^m}{\sqrt{m!}} |0\rangle = |0\,0 \cdots m \cdots\rangle \quad m \geq 0$$

$$\frac{(b_n^{+})^m}{\sqrt{m!}} |0\rangle = |0 \cdots m\rangle \quad m = 0,1$$

For a general state

we have

$$|m_1 m_2 \cdots \rangle =$$

$$\frac{(a_1^\dagger)^{m_1}}{\sqrt{m_1!}} \frac{(a_2^\dagger)^{m_2}}{\sqrt{m_2!}} \cdots \quad |0\rangle$$

$$|m_1 m_2 \quad \rangle =$$

$$\frac{(b_1^\dagger)^{m_1}}{\sqrt{m_1!}} \frac{(b_2^\dagger)^{m_2}}{\sqrt{m_2!}} \cdots \quad |0\rangle$$

where in Fermions $i < j$ implies $b_i^\dagger$ is to the left of $b_j^\dagger$ (this is just a convention

$$\langle m_1 m_2 \cdots | \equiv (|m_1 m_2 \cdots \rangle)^\dagger$$

connection with the "x" representation

$$a_n^\dagger a_m^\dagger |0\rangle \longleftrightarrow \frac{1}{\sqrt{2}}(|\phi_n\rangle|\phi_m\rangle + |\phi_m\rangle|\phi_n\rangle)$$

$$(a_n^\dagger)^2|0\rangle \;-\; |\phi_n\rangle|\phi_n\rangle$$

$$b_n^\dagger b_m^\dagger |0\rangle \;-\; \frac{1}{\sqrt{2}}(|\phi_n\rangle|\phi_m\rangle - |\phi_m\rangle|\phi_n\rangle)$$

The operators a_n a_n^+ b_n b_n^+

are an irreducible set

of operators – this means

that we can express any

operator in terms of these

operators

consider

$$H = \sum_{n=1}^{N} \frac{\vec{p}_n^{\,2}}{2m} + \frac{1}{2} \sum_{n \neq k} V(\vec{r}_n - \vec{r}_k)$$

$\frac{\vec{p}_n^{\,2}}{2m}$ are called 1 body operators

$V(\vec{r}_n - \vec{r}_m)$ is called a two body operator

claim

$$\sum \frac{\vec{p}_n^{\,2}}{2m} = \sum a_\ell^+ \langle \ell | \frac{\vec{p}^{\,2}}{2m} | k \rangle a_k$$

where

$$\langle l | \frac{p^2}{2m} | k \rangle = \int \varphi_l^*(r) \left(-\frac{\hbar^2}{2m}\nabla^2\right) \varphi_k(r) \, d^3r$$

is the matrix element of the operator is the basis states $|\varphi_l\rangle$ and $|\varphi_k\rangle$

To see this consider

$$\langle 0 | a_n \left(\sum_l a_l^\dagger \langle l | \frac{p^2}{2m} | k \rangle a_k \right) a_m^\dagger | 0 \rangle$$

$$\underbrace{a_n a_l^\dagger}_{\delta_{nl}} - a_l^\dagger \underbrace{a_n}_{\downarrow\, 0} \qquad\qquad \underbrace{a_k a_m^\dagger}_{\delta_{mk}} - a_m^\dagger \underbrace{a_k}_{\uparrow\, 0}$$

$$= \langle n | \frac{p^2}{2m} | m \rangle$$

Next consider 2 particle matrix elements of this operator

$$\langle 0 | \underbrace{a_{n_2} a_{n_1} \Big(\sum a_k^{\dagger} \langle k |}_{\delta_{n_1 k} a_{n_2} + \delta_{n_2 k} a_{n_1}} \frac{p^2}{2m} | l \rangle \underbrace{a_l \Big) a_{m_1}^{\dagger}}_{\delta_{l m_1} a_{m_2}^{\dagger} + a_{m_1}^{\dagger} \delta_{l m_2}} a_{m_2}^{\dagger} | 0 \rangle$$

This gives 4 terms

$$\langle n_1 | \frac{p^2}{2m} | m_1 \rangle \delta_{n_2 m_2}$$

$$\langle n_2 | \frac{p^2}{2m} | m_2 \rangle \delta_{n_1 m_1} +$$

$$\langle n_1 | \frac{p^2}{2m} | m_2 \rangle \delta_{n_2 m_1} \quad +$$

$$\langle n_2 | \frac{p^2}{2m} | m_1 \rangle \delta_{n_1 m_2}$$

Some of these terms will be 0 – but if $n_1 = m_1 \neq n_2 = m_2$ we get

$$\langle n_1 | \frac{p^2}{2m} | n_1 \rangle + \langle n_2 | \frac{p^2}{2m} | n_2 \rangle$$

For 2 particle operators like 2 body potentials (here I illustrate the case of Fermions)

$$V = \frac{1}{2} \sum b^{+}_{m_1} b^{+}_{m_2} \langle m_1 m_2 | V | n_1 n_2 \rangle b_{n_2} b_{n_1}$$

* here the ordering is consistent with the convention

$$|n_1 n_2\rangle = b^{+}_{n_1} b^{+}_{n_2} |0\rangle$$

$$\langle m_1 m_2| = \langle 0| b_{m_2} b_{m_1}$$

$$\langle 0| b_{k_2} b_{k_1} V b^{+}_{\ell_1} b^{+}_{\ell_2} |0\rangle =$$

$$\frac{1}{2} \sum_{mn} \underbrace{\langle 0| b_{k_2} b_{k_1} b^{+}_{m_1} b^{+}_{m_2}}_{\delta_{k_1 m_1} \delta_{k_2 m_2} - \delta_{k_1 m_2} \delta_{k_2 m_1}} \langle m_1 m_2 | V | n_1 n_2 \rangle \underbrace{b_{n_2} b_{n_1} b^{+}_{\ell_1} b^{+}_{\ell_2} |0\rangle}_{\delta_{n_1 \ell_1} \delta_{n_2 \ell_2} - \delta_{n_1 \ell_2} \delta_{n_2 \ell_1}}$$

$$\frac{1}{2}\Big(\langle k_1 k_2 | V | l_1 l_2 \rangle +$$
$$\langle k_2 k_1 | V | l_2 l_1 \rangle -$$
$$\langle k_1 k_2 | V | l_2 l_1 \rangle -$$
$$\langle k_2 k_1 | V | l_1 l_2 \rangle \Big)$$

the first 2 terms and last 2 terms are identical

$$\langle k_1 k_2 | V | l_1 l_2 \rangle - \langle k_2 k_1 | V | l_1 l_1 \rangle$$

If these were bosons the term with the - sign would be added rather than subtracted

Finally note if these we 2 bosons in the same state

$$\underbrace{\langle 0 | \frac{(a_k)^2}{\sqrt{2!}} \sum \frac{1}{2} a_{m_2}^{+} a_{m_1}^{+}}_{\frac{2}{\sqrt{2}} \delta_{m_1 k} \delta_{m_1 k} \frac{1}{2}} \langle m_1 m_2 | V | n_1 n_2 \rangle \underbrace{a_{n_2} a_{n_1} \frac{(a_l^{+})^2}{\sqrt{2}}}_{\frac{2}{\sqrt{2}} \delta_{n_1 l} \delta_{n_2 l}} | 0 \rangle$$

$= \langle kk | V | ee \rangle$

We can combine these to write a general Hamiltonian with 2 body interaction (fermions)

$$H = \sum_{ke} b_k^\dagger \langle k | \frac{p^3}{2m} | e \rangle b_e + \frac{1}{2} \sum b_{k_1}^\dagger b_{k_2}^\dagger \langle k_1 k_2 | V | n_1 n_2 \rangle b_{n_2} b_{n_1}$$

To use this for $N = 3$ particle system

$$|\Psi\rangle = \sum b_{k_1}^\dagger b_{k_2}^\dagger b_{k_3}^\dagger |0\rangle \, C(k_1 k_2 k_3)$$

here we dont worry about namalizatn

We want to

$$H \sum b^+_{k_1} b^+_{k_2} b^+_{k_3} |0\rangle C_{k_1k_2k_3} =$$

$$E \sum b^+_{k_1} b^+_{k_2} b^+_{k_3} |0\rangle C_{k_1k_2k_3}$$

When H is applied to $b^+_{k_1} b^+_{k_2} b^+_{k_3}$ it will replace some of the k_i's by a different set of k_i — there will still be a product of 3 b^+_k's. — equations coefficients of independent $b^+_{k_1} b^+_{k_2} b^+_{k_3} |0\rangle$ (using the antisymmetry of $C_{k_1k_2k_3}$) give a set of coupled equations for the coefficients $C_{k_1k_2k_3}$

Fields

It is useful to define

$$\psi(x) = \sum_{n=1}^{\infty} \langle x | \phi_n \rangle a_n$$

$$\psi^{+}(x) = \sum_{n=1} a_n^{+} \langle \phi_n | x \rangle$$

These are operators - called field operators. - we also have similar expression for Fermi fields

$$\psi(x) = \sum \langle x | \phi_n \rangle b_n$$

$$\psi^{+}(x) - \sum b_n^{+} \langle \phi_n | x \rangle$$

Note that these operators satisfy

$$[\psi(\bar{x}), \psi^{+}(\bar{y})] =$$

$$\sum_{mn} \langle \bar{x} | \phi_n \rangle \underbrace{[a_n\, a_m^{+}]}_{\delta_{nm}} \langle \phi_m | \bar{y} \rangle$$

$$\sum_{mn} \langle \bar{x} | \phi_n \rangle \langle \phi_n | \bar{y} \rangle = \delta(\bar{x} - \bar{y})$$

$$\{\psi(\bar{x}), \psi^{+}(\bar{y})\} =$$

$$\sum \langle x | \phi_n \rangle \underbrace{\{b_n\, b_m^{+}\}}_{\delta_{nm}} \langle \phi_m | \bar{y} \rangle$$

$$\sum \langle \bar{x} | \phi_n \rangle \langle \phi_n | \bar{y} \rangle = \delta(\bar{x} - \bar{y})$$

We can express operators in terms of fields.

To do this note

$$\psi(x) = \sum \langle x | n \rangle a_n \Rightarrow$$

$$\int \langle m | x \rangle \psi(x)\, dx = \sum_n \int \langle m | x \rangle \langle x | n \rangle a_n$$

$$= \sum_n \delta_{mn} a_n = a_m$$

$$a_m = \int \langle m|x\rangle \psi(x)$$

$$a_n^\dagger = \int \psi^\dagger(x) \langle x|n\rangle$$

using these relations gives

$$\sum a_n^\dagger \langle n|Q|m\rangle a_m =$$

$$\sum \int dx\, dy\, \psi^\dagger(x) \langle x\underbrace{|n\rangle\langle n|}_{I}Q\underbrace{|m\rangle\langle m|}_{I}y\rangle \psi(y)$$

$$\int \psi^\dagger(x) \langle x|Q|y\rangle \psi(y)\, dx\, dy$$

(these are operators - not wave functions)

Note the dependence on the original basis has disappeared

$$V = \frac{1}{2} \sum b^{\dagger}_{n_1} b^{\dagger}_{n_2} \langle n_1 n_2 | V | m_1 m_2 \rangle \, b_{m_2} b_{m_1}$$

$$= \int \frac{1}{2} \psi^{\dagger}(x_1) \psi^{\dagger}(x_2) \langle x_1 | n_1 \rangle \langle x_2 | n_2 \rangle \langle n_1 n_2 | V | m_1 m_2 \rangle \langle m_1 | y_1 \rangle \langle m_2 | y_2 \rangle \, \psi(y_2) \psi(y_1)$$

using completeness of the single particle states

$$V = \frac{1}{2} \int \psi^{\dagger}(x_1) \psi^{\dagger}(x_2) \langle x_1 x_2 | V | y_1 y_2 \rangle \, \psi(y_2) \psi(y_1) \, d^3x_1 \, d^3x_2 \, d^3y_1 \, d^3y_2$$

Again - the dependence on the basis has disappeared (actually this is equivalent to switching to a coordinate basis

$$H = \int d\bar{x}_1 d\bar{x}_2 \; \psi^\dagger(x_1) \langle x_1 | \frac{p^2}{2m} | x_2 \rangle \psi(x_2)$$

$$+ \frac{1}{2} \int dx_1 dx_2 dy_1 dy_2 \; \psi^\dagger(x_1) \psi^\dagger(x_2)$$

$$\langle x_1 x_2 | V | y_1 y_2 \rangle \psi(y_2) \psi(y_1))$$

It this case a general 2 body state can be expressed as

$$|\psi\rangle = \int \psi^\dagger(x) \psi^\dagger(y) |0\rangle f(xy) \, dx \, dy$$

Then we have

$$H|\psi\rangle = \int dx_1 dx_2 dx dy \; \times$$

$$\langle x_1 | \frac{p^2}{2m} | x_2 \rangle \psi^\dagger(x_1) \underbrace{\psi(x_2) \psi^\dagger(x) \psi^\dagger(y)}_{\substack{\delta(x_2 - x) \psi^\dagger(y) \\ \pm \delta(x_2 - y) \psi^\dagger(x)}} |0\rangle$$

$$\times f(xy) +$$

$$\frac{1}{2}\int \psi^{+}(x_1)\,\psi^{+}(x_2)\,\langle x_1 x_2 | V | y_1 y_2 \rangle$$

$$\underbrace{\psi(y_2)\,\psi(y_1)\,\psi^{+}(x)\,\psi^{+}(y)\,|0\rangle}_{\substack{\delta(y_1-x)\,\delta(y_2-y)\,\pm \\ \delta(y_1-y)\,\delta(y_2-x)}}\, f(xy)$$

$$\psi^{+}(x_1)\,\langle x_1 | Q | x \rangle\,\psi^{+}(x_2)\,f(x x_2)\;dx_1\,dx_2\,dx$$

$$\psi^{+}(x_1)\,\langle x_1 | Q | x \rangle\,\psi^{+}(x_2)\,f(x_2 x)\;dx_1\,dx_2\,dx$$

$$\frac{1}{2}\,\psi^{+}(x_1)\,\psi^{+}(x_2)\,\langle x_1 x_2 | V | x y \rangle$$

$$\left(f(xy) \pm f(yx)\right)\,dx_1\,dx_2\,dx\,dy$$

$$= \quad E\int \psi^{+}(x_1)\,\psi^{+}(x_2)\,|0\rangle\, f(x_1, x_2)$$

It is also possible to use

$$\frac{dF}{dt} = \frac{i}{\hbar}\,[H\,F]$$

to find the time dependence of an operator F