Lecture 40

Identical particles

Last time

To get the correct probabilistic interpretation of quantum mechanics there must be a 1-1 correspondence between physical states and rays in the Hilbert space

Let T_{ij} be the operator that interchanges particles i and j. We showed

$$T_{ij}^2 = I \qquad T_{ij} = T_{ij}^{+}$$

If i and j are identical particles then the correspondence requires

$$T_{ij}|\phi\rangle = |\phi\rangle e^{i\phi}$$

$$T_{ij}^2|\phi\rangle = |\phi\rangle = |\phi\rangle e^{2i\phi}$$

This requires $\phi = 0, \pi$ which means

$$T_{ij}|\phi\rangle = \pm|\phi\rangle$$

We found a contradiction if on a given physical state some T_{ij} had $+1$ eigenvalues and others had -1 eigenvalues

$\therefore$ A state representing a system of identical particles must be either symmetric or antisymmetric with respect to interchanging identical particles

Symmetrization postulate

states of half integer spin identical particles are antisymmetric (Fermions)

states of integer spin identical particles are symmetric (Bosonic)

Permutation group

$i \neq j \quad \sigma(i) \neq \sigma(j)$

$$
\begin{aligned}
1 &\to \sigma(1) \\
2 &\to \sigma(2) \\
&\vdots \\
N &\to \sigma(N)
\end{aligned}
$$

properties

① $N!$ permutations for N objects (particles)

(2) products

$$\sigma_2 \cdot \sigma_1 (i) = \sigma_2(\sigma_1(i))$$

(3) every permutation is the product of pairwise transpositions

(4) every permutation is a product of either an even or odd # of transpositions

(5) $|\sigma| = 1$ even # transposition

$|\sigma| = -1$ odd # transposition

(6) $\sigma_i \sigma_j \neq \sigma_j \sigma_i$

the product of permutations does not generally commute

(7) The inverse of a permutation is the product of the transpositions in reverse order

So $|\sigma| = |\sigma^{-1}|$

(8) $\sum_{\sigma} f(\sigma) = \sum_{\sigma} f(\sigma'(\sigma))$

(9) Let σ_i = even permutation

pick $\sigma_e \Rightarrow \sigma_e \sigma_i$ are odd permutation $\Rightarrow$

$\frac{N!}{2}$ even permutation

$\frac{N!}{2}$ odd permutations

Projectors on symmetric

or antisymmetric subspace

of N particle Hilbert space

P_σ = permutation operator

$\mathcal{P}(N)$ = set of permutations

on N objects

$$S \equiv \frac{1}{N!} \sum_{\sigma \in \mathcal{P}(N)} P_\sigma$$

$$A = \frac{1}{N!} \sum_{\sigma \in \mathcal{P}(N)} (-)^{|\sigma|} P_\sigma$$

Note

$$S^2 = \frac{1}{(N!)^2} \sum_{\sigma, \sigma' \in \mathcal{P}(N)} P_\sigma P_{\sigma'} =$$

$$= \frac{1}{(N!)^2} \sum_{\sigma, \sigma' \in \mathcal{P}(N)} P_{\sigma\sigma'}$$

Let $\sigma'' = \sigma\sigma'$ $\sum_{\sigma'} = \sum_{\sigma''}$ $\Rightarrow$

$$= \frac{1}{(N!)^2} \sum_{\sigma} \left(\sum_{\sigma'' \in \mathcal{P}(N)} P_{\sigma''} \right) =$$

$$\frac{1}{N!}\sum_{\sigma} \cdot S = \frac{N!}{N!} S = S$$

$$\therefore \quad S^2 = S$$

Also not

$$S^{+} = \frac{1}{N!}\sum_{\sigma \in P(N)} P_{\sigma}^{+} =$$

$$\frac{1}{N!}\sum_{\sigma \in P(N)} P_{\sigma^{-1}} =$$

Let $\sigma'' = \sigma^{-1}$

$$\frac{1}{N!}\sum_{\sigma'' \in P(N)} P_{\sigma''} = S$$

$$\therefore \quad S = S^{+}$$

similarly

$$A^2 = \frac{1}{(N!)^2}\sum_{\sigma\sigma'} (-)^{|\sigma|+|\sigma'|} P_{\sigma} P_{\sigma'} =$$

$$A^2 = \frac{1}{(N!)^2}\sum_{\sigma'\sigma} (-)^{|\sigma+\sigma'|} P_{\sigma\sigma'}$$

Let $\sigma'' = \sigma\sigma'$ $\quad (-)^{|\sigma''|} = (-)^{|\sigma|+|\sigma'|}$

$$= \frac{1}{(N!)^2} \sum_{\sigma} \sum_{\sigma''} (-1)^{|\sigma''|} P_{\sigma''}$$

$$= \frac{1}{N!} \sum_{\sigma} A = \frac{N!}{N!} A$$

similarly $A = A^{\dagger}$ uses the same argument used for S with $|\sigma'| = |\sigma|$ since they both have the same # of transpositions

Finally note

$$AS = \frac{1}{(N!)^2} \sum (-1)^{\sigma} P_{\sigma} P_{\sigma'} =$$

$$\frac{1}{(N!)^2} \sum (-1)^{|\sigma|} P_{\sigma\sigma'} \qquad \text{let } \sigma'' = \sigma\sigma'$$

$$\frac{1}{(N!)} \sum (-)^{|\sigma|} \frac{1}{N!} \sum_{\sigma''} P_{\sigma''} =$$

$$\frac{1}{N!} \sum (-1)^{|\sigma|} S$$

but $\sum_i (-1)^{|\sigma|} = 0$ because there are $\frac{N!}{2}$ even and $\frac{N!}{2}$ odd permutations.

$\therefore$ S and A are orthogonal projection operators

If H is a Hamiltonian for a system of identical particles

$$[S\ H] = [A\ H] = 0$$

the subspaces do not mix - it has symmetric and antisymmetric eigenstates, but only one is physically relevant depending on the spin of the particles

Normalization

$|\psi\rangle \to A|\psi\rangle$ or $S|\psi\rangle$

~~If~~ $|\psi\rangle$ is not symmetric or antisymmetric

$$|\hat{\psi}\rangle = \frac{S|\psi\rangle}{\langle\psi|S^\dagger S|\psi\rangle^{1/2}} = \frac{S|\psi\rangle}{\langle\psi|S^2|\psi\rangle^{1/2}}$$

$$= \frac{S|\psi\rangle}{\langle\psi|S|\psi\rangle^{1/2}}$$

similarly

$$|\hat{\psi}\rangle = \frac{A|\psi\rangle}{\langle\psi|A|\psi\rangle^{1/2}}$$

Also note

$\langle\phi|H|S\psi\rangle =$

$\langle\phi|H\,S^2|\psi\rangle =$

$\langle\phi|SHS|\psi\rangle =$

$\langle\phi|S^\dagger H\,S|\psi\rangle$

similarly

$\langle\phi|H|A|\psi\rangle =$

$\langle\phi|A^\dagger HA|\psi\rangle$

which means that it is sufficient to symmetrize or antisymmetrize only the initial & final states (although they must be normalized.)

Occupation # representation

① start with an orthonormal basis of single particle states

$\{|\phi_n\rangle\}_{n=1}^{\infty}$

$\langle\phi_n|\phi_m\rangle = \delta_{nm}$

② products on N such states are a basis for the N particle Hilbert space

$$|\psi\rangle = \sum c_{n_1 \, n_\infty} |\phi_{n_1} \; \phi_{n_\infty}\rangle$$

③ when the particles are identical all we know (up to sign) is how many particles are in state1, state 2, ...

④ introduce a new representation called the occupation # representation

$$|m_1 m_2 \cdots \qquad m_\infty\rangle$$

m_1 identical particles in state $|\phi_1\rangle$

m_2 identical particles in state $|\phi_2\rangle$

$\vdots$

m_n identical particles in state $|\varphi_n\rangle$

$1 \le n \le \infty$

This representation can be used to describe systems with any number of identical particles

Note normally all but a finite number of the m_i are 0

$$N = \sum_{k=1}^{\infty} m_k$$

For Fermions $m_k = 0$ or 1

For Bosons m_k can be any non negative integer

In order to use this representation we define

$$|0\rangle = |0\,0\,0\,0 \cdots \rangle$$

the state consisting of no particles. We also define

$$\langle 0|0\rangle \equiv 1$$

Next we define non Hermitian operators

$a_n \quad a_n^+$ for Bosons

$b_n \quad b_n^+$ for Fermions

* a_n^+ adds 1 particle in state $|\phi_n\rangle$

a_n removes 1 particle in state $|\phi_n\rangle$

b_n^+ add 1 Fermion in state $|\phi_n\rangle$

b_n removes 1 Fermion in state $|\phi_n\rangle$

we require

$$[a_n\, a_m^+] = \delta_{mn}$$

$$\{b_n\, b_m^+\} = \delta_{mn}$$

$$[a_n\, a_m] = [a_n^+\, a_m^+] = 0$$

$$\{b_n\, b_m\} = \{b_n^+\, b_m^+\} = 0$$

For Fermion:

$$b_n^+ b_m^+ = -b_m^+ b_n^+$$

so interchanging identical particles changes the sign of the state an

$$b_n^+ b_n^+ = -b_n^+ b_n^+ = 0$$

so we cannot to 2 fermions in the same single particle state

ordering convention

$$(a_1^\dagger)^{m_1} (a_2^\dagger)^{m_2} \cdots |0\rangle \equiv |m_1 m_2 m_3 \cdots\rangle N$$

$$(b_1^\dagger)^{m_1} (b_2^\dagger)^{m_2} \cdots |0\rangle$$
$$|m_1 m_2 m_3 \cdots\rangle N$$

where N is a normalization constant that must be determined

In order to determine the normalization note

$$N_m \equiv a_m^+ a_m$$

$$[N_m, a_m] = a_m^+ a_m a_m - a_m a_m^+ a_m$$

$$= [a_m^+, a_m] a_m$$

$$= -a_m$$

$$[N_m, a_m^+] = a_m^+ a_m a_m^+ - a_m^+ a_m^+ a_m$$

$$= a_m^+ [a_m a_m^+]$$

$$= a_m^+$$

$$N_m \equiv b_m^+ b_m$$

$$[N_m b_m] = b_m^+ b_m b_m + b_m b_m^+ b_m$$

$$= \{b_m^+ b_m\} b_m$$

$$= b_m$$

$$[N_m b_m^+] = b_m^+ b_m b_m^+ + b_m^+ b_m^+ b_m$$

$$= b_m^+ \{b_m b_m^+\}$$

$$= b_m^+$$

consider

$N_k (a_k^\dagger)^m |0\rangle =$

$N_k a_k^\dagger (a_k^\dagger)^{m-1} |0\rangle$

$a_k^\dagger (N_k+1)(a_k^\dagger)^{m-1} |0\rangle$

$(a_k^\dagger)^2 (N_k+2)(a_k^\dagger)^{m-2} |0\rangle$ ---

$(a_k^\dagger)^N (0+N_k) |0\rangle$

$N_k (a_k^\dagger)^m |0\rangle = m \, |(a_k^\dagger)^m |0\rangle$

can also be done by

induction - assume

$N_k (a_k^\dagger)^n |0\rangle = n \, (a_k^\dagger)^n |0\rangle$

$N_k (a_k^\dagger)^{n+1} |0\rangle =$

$(N_k a_k^\dagger - a_k^\dagger N_k + a_k^\dagger N_k)(a^\dagger)^n |0\rangle$

$a_k^\dagger (1+n) |(a_k^\dagger)^n |0\rangle =$

$(n+1)(a_k^\dagger)^{n+1} |0\rangle$

similarly for b_k

$$b_k^{\dagger} |0\rangle_k = |1_k\rangle$$

$$b_k^{\dagger} |1_k\rangle = 0$$

Normalization

$$a_k^{\dagger} |m_k\rangle = |m_{k+1}\rangle c$$

$$c^2 = \langle m_k | a_k a_k^{\dagger} | m_k\rangle$$

$$= \langle m_k | a_k a_k^{\dagger} - a_k^{\dagger} a_k + a_k^{\dagger} a_k | m_k\rangle$$

$$= \langle m_k | (1 + N_n) | m_k\rangle$$

$$= (m_{k+1}) \langle m_k | m_k\rangle$$

$$= m_{k+1}$$

$$\therefore \quad c = \sqrt{m_k + 1}$$

$$a_k^{\dagger} | m_k\rangle = |m_{k+1}\rangle \sqrt{m_k + 1}$$

$$(a_k^{\dagger})^{n_k} |0\rangle = \sqrt{n_k!}\, |n_k\rangle$$

$|m_1 \quad m_n \ldots\rangle =$

$$\prod \frac{(a_n^\dagger)^{m_n}}{\sqrt{m_n!}} |0\rangle$$

we can do the same

for fermions – we can't

have $m_n = 1$ or 0 for 1

we g

$$\prod \frac{(b_n^\dagger)^{m_n}}{\sqrt{m_n}} |0\rangle$$

although in this case

$$\prod \frac{(b_n^\dagger)^{r}}{\sqrt{m_n}} = \frac{(b_1^\dagger)^{m_1}}{\sqrt{m_1!}} \frac{(b_2^\dagger)^{m_2}}{\sqrt{m_2!}} \cdots \quad |0\rangle$$

where the order matters

conventn

$$|m_1 \cdots \rangle = \frac{(a_1^\dagger)^{m_1}}{\sqrt{m_1!}} \frac{(a_2^\dagger)^{m_2}}{\sqrt{m_2!}} \cdots |0\rangle$$

$$\langle m_1 m_2 \quad | = \langle 0| \cdots \frac{(a_2)^{m_2}}{\sqrt{m_2!}} \frac{(a_1)^{m_1}}{\sqrt{m_1!}}$$

$$|m_1 \quad \rangle = \frac{(b_1^\dagger)^{m_1}}{\sqrt{m_1!}} \frac{(b_2^\dagger)^{m_2}}{\sqrt{m_2!}} \cdots |0\rangle$$

$$\langle m_1 m_2 \quad | = \langle 0| \cdots \frac{(b_2)^{m_2}}{\sqrt{m_2!}} \frac{(b_1)^{m_1}}{\sqrt{m_1!}}$$

Operatrs

1 Body operatr

$$O = \sum_{mn} a_m^\dagger \langle m|O|n\rangle a_n$$

$$\langle m|O|n\rangle =$$

$$\int \phi_m^*(\bar r) \langle \bar r|O|\bar r'\rangle \phi_n(\bar r')\, d^3r\, d^3r'$$

consider

$$\langle 1_1\, 2_2 \,|\, \Theta \,|\, 2_3\, 1_2 \rangle$$

$$\sum_{mn} \langle 0| \frac{(a_2)^2}{\sqrt{2!}} \frac{a_1}{\sqrt{1!}}\, a_m^\dagger \langle m|\Theta|n\rangle a_n \times$$

$$\frac{(a_3^\dagger)^2}{\sqrt{2!}} \frac{a_2^\dagger}{\sqrt{1!}} |0\rangle$$

remark

$$a_n (a_n^\dagger)^m = m (a_n^\dagger)^{m-1}$$

$$(a_n)^m a_n^\dagger = m (a_n)^{m-1}$$

this matrix element

$$\langle 0| \frac{a_2 a_1}{\sqrt{2!}}\, 2 \langle 2|\Theta|2\rangle \frac{(a_3^\dagger)^2}{\sqrt{2!}} |0\rangle +$$

$$\langle 0| \frac{(a_2)^2}{\sqrt{2}}\, 1 \langle 1|\Theta|2\rangle \frac{(a_3^\dagger)^2}{\sqrt{2!}} |0\rangle +$$

$$\langle 0| \frac{a_2 a_1}{\sqrt{2}}\, 2 \langle 2|\Theta|3\rangle \frac{2}{\sqrt{2!}}\, a_3^\dagger a_1^\dagger |0\rangle$$

$$\langle 0| \frac{(a_2)^2}{\sqrt{2}}\, 1 \langle 1|\Theta|3\rangle \frac{2}{\sqrt{2}}\, a_3^\dagger a_1 |0\rangle$$

This vanishes - to get something non zero all states have to be the same except one initial and one final state

$$\langle 0| a_3 a_2 a_1 \sum_n a_n^+ \langle n|Q|m\rangle a_m a_1^+ a_2^+ a_3^+ \rangle$$

$$\langle 1|Q|1\rangle + \langle 2|Q|2\rangle + \langle 3|Q|3\rangle$$

which gives the sum of all 3 single particle matrix elements

$$\langle 0| \frac{a_3^2}{\sqrt{2}} a_2 a_1 \sum a_n^+ \langle n|Q|m\rangle a_1 a_2 \frac{a_3^+}{\sqrt{2}} |-\rangle$$

$$= \frac{1}{2} \cdot 2 \cdot 2 \langle 3|Q|3\rangle + \langle 1|Q|1\rangle + \langle 2|Q|2\rangle$$

$$= 2\langle 3|Q|3\rangle + \langle 1|Q|1\rangle + \langle 2|Q|2\rangle$$

Next we define 2 body operators - first note

$$|n_1 n_2\rangle = \frac{1}{\sqrt{2}}(|n_1\rangle|n_2\rangle + |n_2 n_1\rangle)$$

for bosons

$$|n_1 n_2\rangle = \frac{1}{\sqrt{2}}(|n_1\rangle|n_2\rangle - |n_2 n_1\rangle)$$

for fermions

$$\langle n_1 n_2|V|n_1 n_2\rangle =$$

$$\frac{1}{2}(\langle n_1|\langle n_2| \pm \langle n_1|\langle n_2|)V(|n_1\rangle|n_2\rangle \pm |n_1 n_2\rangle)$$

We define

$$\hat{V} = \frac{1}{2}\sum a^{+}_{n_1} a^{+}_{n_2} \langle n_1 n_2|V|m_1 m_2\rangle$$
$$a_{m_2} a_{m_1}$$

consider the expectation value of this operator in a 2 particle state

$$\langle 0 | a_2 a_1 V | a_1^\dagger a_2^\dagger | 0 \rangle =$$

$$\frac{1}{2} \sum \langle 0 | a_2 a_1 a_{n_1}^\dagger a_{n_2}^\dagger \langle n_1 n_2 | V | m_1 m_2 \rangle$$

$$a_{m_2} a_{m_1} | a_1^\dagger a_2^\dagger | 0 \rangle =$$

$$\frac{1}{2} \langle 12 | V | 12 \rangle +$$

$$\frac{1}{2} \langle 21 | V | 21 \rangle +$$

$$\frac{1}{2} \langle 12 | V | 21 \rangle +$$

$$\frac{1}{2} \langle 21 | V | 12 \rangle =$$

$$\langle 12 | V | 12 \rangle + \langle 12 | V | 21 \rangle$$

since

$$\langle 12 | V | 12 \rangle = \langle 21 | V | 21 \rangle$$

$$\langle 12 | V | 21 \rangle = \langle 21 | V | 12 \rangle$$

If these were fermions the $\langle 12|V|21\rangle$ and $\langle 21|V|12\rangle$ terms would have $-$ sign