Lecture 3

$$\vec{J}^2 |j\mu\rangle = j(j+1)|j\mu\rangle$$
$$J_z |j\mu\rangle = \mu |j\mu\rangle$$

$$J_\pm = J_x \pm i J_y$$

$$J_\pm |j\mu\rangle = |j\,\mu\pm 1\rangle \sqrt{(j\mp\mu)(j\pm\mu+1)} \qquad -j \le \mu \le j$$
$$|j\,\mu\pm 1\rangle \sqrt{j(j+1)\mp\mu(\mu\pm 1)} \qquad j = \frac{n}{2} \quad n = 0,1,2\cdots$$

$$\mathcal{U}(R) = e^{-i\vec{J}\cdot\vec{\Theta}} \qquad \hat{\Theta} = \text{axis of rotation}$$
$$|\vec{\Theta}| = \text{angle of rotation}$$

$$\langle j\mu|\mathcal{U}(R)|j'\nu\rangle = D^j_{\mu\nu}(R)\,\delta_{jj'}$$

$$\sum_\mu D^j_{\mu\nu}(R_2)\, D^j_{\nu\alpha}(R_1) = D^j_{\mu\alpha}(R)$$

$$D^j_{\mu\nu}(R)^\dagger = D^j_{\nu\mu}(R)^* = D^j_{\mu\nu}(R)^{-1} = D^j_{\mu\nu}(R^{-1})$$

$$D^j_{\mu\nu}(I) = \delta_{\mu\nu}$$

$2j+1$ dimensional representation of $SU(2)$

(also called Wigner rotation matrices)

computation of $D^j_{\mu\nu}(R)$

Schwinger method

$$n_\pm = j \pm \mu$$

$$j = \tfrac{1}{2}(n_+ + n_-)$$
$$j = \tfrac{1}{2}(n_+ - n_-)$$

change notation

$|n_+ n_-\rangle = |j\, \mu\rangle$

define operators

$$a_+ |n_+ n_-\rangle = |n_+ - 1, n_-\rangle \sqrt{n_+}$$

$$a_- |n_+ n_-\rangle = |n_+\, n_- - 1\rangle \sqrt{n_+}$$

note

$$\langle n_+' n_-' | a_+ | n_+ n_-\rangle = \langle n_+ n_- | a_+^\dagger | n_+' n_-'\rangle^* =$$

$$\langle n_+' n_-' | n_+ - 1\, n_-\rangle \sqrt{n_+} = \sqrt{n_+' + 1}\, \delta_{n_-' n_-}\, \delta_{n_+, n_+' + 1}$$

$$a_+^\dagger | n_+ n_-\rangle = |n_+ + 1\, n_-\rangle \sqrt{n_+ + 1}$$

$$a_-^\dagger | n_+ n_-\rangle = |n_+\, n_- + 1\rangle \sqrt{n_- + 1}$$

homework

$$[a_+\, a_+^\dagger] = [a_-\, a_-^\dagger] = 1$$

all other commutators vanish

$$J_\pm |n_+ n_-\rangle = |j\, \mu \pm 1\rangle \sqrt{(j \mp \mu)(j \pm \mu + 1)}$$

$$n_+ = j + \mu \pm 1 = n_+ \pm 1 \qquad n_\mp \qquad n_\pm + 1$$

$$n_- = j - (\mu \pm 1) = n_- \mp 1$$

$$J_\pm |n_+ n_-\rangle = |n_+ \pm 1\, n_- \mp 1\rangle \sqrt{n_\mp (n_\pm + 1)}$$

$$\boxed{J_\pm |n_+ n_-\rangle = a_\pm^\dagger a_\mp |n_+ n_-\rangle}$$

Homework

$$\boxed{J_z |n_+ n_-\rangle = \frac{1}{2}(a_+^\dagger a_+ - a_-^\dagger a_-)|n_+ n_-\rangle}$$

These can be summarized using the notation

$$a^\dagger = (a_+^\dagger \; a_-^\dagger)$$

$$a = \begin{pmatrix} a_+ \\ a_- \end{pmatrix}$$

$$J_z = \frac{1}{2}(a_+^\dagger \; a_-^\dagger)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_+ \\ a_- \end{pmatrix}$$

$$J_\pm = (a_+^\dagger \; a_-^\dagger)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_+ \\ a_- \end{pmatrix}$$

$$J_x = \frac{1}{2}(J_+ + J_-) = \frac{1}{2}(a_+^\dagger \; a_-^\dagger)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_+ \\ a_- \end{pmatrix}$$

$$J_y = \frac{i}{2}(J_- - iJ_+) = \frac{1}{2}(a_+^\dagger \; a_-^\dagger)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_+ \\ a_- \end{pmatrix}$$

$$\boxed{\vec{J} = \frac{1}{2} a^\dagger \vec{\sigma} a}$$

From the definition

$$|n_+ n_-\rangle = \frac{(a_+^\dagger)^{n_+}}{\sqrt{n_+!}} \frac{(a_-^\dagger)^{n_-}}{\sqrt{n_-}}$$

using this we get

$$D^j_{\mu\nu}(R) = \langle 00| \frac{(a_+)^{n_+'}}{\sqrt{n_+'!}} \frac{(a_-)^{n_-'}}{\sqrt{n_-'!}} \underbrace{e^{-i\frac{1}{2} a^\dagger \bar{\sigma} a}}_{U(R)} \frac{(a_+^\dagger)^{n_+}}{\sqrt{n_+!}} \frac{(a_-^\dagger)^{n_-}}{\sqrt{n_-!}} |00\rangle$$

step 1

$$\mathcal{U} \frac{(a_+^\dagger)^{n_+}}{\sqrt{n_+!}} \frac{(a_-^\dagger)^{n_-}}{\sqrt{n_-!}} |00\rangle =$$

$$\frac{(\mathcal{U}(a_+^\dagger)\mathcal{U}^\dagger)^{n_+}}{\sqrt{n_+!}} \frac{(\mathcal{U}(a_-^\dagger)\mathcal{U}^\dagger)^{n_-}}{\sqrt{n_-!}} \mathcal{U}|00\rangle$$

step 2 note

$$\mathcal{U}|00\rangle = \left(1 + \sum_{n=1}^{\infty} \left(\frac{-i}{2}\theta\right)^n \frac{1}{n!} (a^\dagger \bar{\sigma} a)^n\right)|00\rangle$$

$$= |00\rangle$$

since $a|00\rangle = 0$

step 3 $\mathcal{U} = e^{-iG}$

$$e^{-iG} a_\pm^\dagger e^{iG} = a_\pm^\dagger - i[G, a_\pm^\dagger] + \frac{1}{2!}(-i)^2 [G,[G,a_\pm^\dagger]] + \cdots$$

proof

$$F(\lambda) = U(\lambda)\, X\, U^{+}(\lambda) \qquad U = e^{-i\lambda A}$$

$$\frac{dF}{d\lambda} = \frac{dU}{d\lambda} X U^{+}(\lambda) + U(\lambda) X \frac{d}{d\lambda} U^{+}(\lambda)$$

$$= -iA\, U X U^{+} + U X U^{+} (iA)$$

$$= [-iA, F]$$

$$\frac{d^n F}{d\lambda^n} = [-iA[-iA[\cdots, F]]]$$

Taylor series about 0

$$F(\lambda) = X + \sum_{n=1}^{\infty} \frac{(-i\lambda)^n}{n!} \underbrace{[A[A[A \cdots \quad [A}_{n \text{ times}}, F] \cdots]]]$$

set $\lambda = 1$ gives result

step 4 $\quad G = \frac{1}{2} a^{+} \bar{\sigma} a$

$$[G, a_{\pm}^{+}] = \frac{1}{2} \sum_{ij} [a_i^{+} \sigma_{ij} a_j, a_{\pm}^{+}]$$

$$= \frac{1}{2} \sum_i a_i^{+} \sigma_{i\pm}$$

$$[G, [G, a_{\pm}^{+}]] = [\tfrac{1}{2} \sum a_i^{+} \sigma_{ij} a_j, \tfrac{1}{2} \sum a_e^{+} \sigma_{e\pm}]$$

$$= \left(\tfrac{1}{2}\right)^2 \sum a_i^{+} \sigma_{i\pm}^2$$

$$\underbrace{[G[\cdots}_{n \text{ times}} a_{\pm}^{+}]]] = \left(\tfrac{1}{2}\right)^n \sum a_i^{+} \sigma_{i\pm}^n$$

$$e^{-\frac{i}{2}\theta a^\dagger \bar{\sigma} a}\; a_\pm^\dagger \; e^{\frac{i}{2}\theta a^\dagger \bar{\sigma} a} =$$

$$\sum_{n=}^{\infty} \left(-\frac{i\theta}{2}\right)^n \frac{1}{n!} \sum a_i^\dagger (\sigma^n)_{i+} =$$

$$a_\pm^\dagger \cos\left(\frac{\theta}{2}\right) - \frac{i}{2} \sin\left(\frac{\theta}{2}\right) \sum a_i^\dagger \sigma_{i+} =$$

$$\sum_i a_i^\dagger R_{i\pm}$$

where $R_{ij} = \cos\left(\frac{\theta}{2}\right)\delta_{ij} - i \sin \theta/2 \; \sigma_{ij}$

step 5 insert these in the expression for $D^j_{\mu\nu}(R)$

$$D^j_{\mu\nu}(R) = \sum_{ij} \frac{1}{\sqrt{n_+'! n_-'! n_+! n_-!}} \langle 00|(a_+)^{n_+'}(a_-)^{n_-'}$$

$$(a_+^\dagger R_{++} + a_-^\dagger R_{-+})^{n_+}(a_+^\dagger R_{+-} + a_-^\dagger R_{--})^{n_-}|00\rangle$$

step 6 use the binomial theorem

$$(a+b)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^k b^{n-k}$$

$$(a_+^\dagger R_{+\pm} + a_-^\dagger R_{-\pm})^{n_\pm} =$$

$$\sum_{k=0}^{n_\pm} \frac{(n_\pm)!}{k!(n_\pm-k)!} (a_+^\dagger)^k (R_{+\pm})^k (a_-^\dagger)^{n_\pm-k} (R_{-\pm})^{n_\pm-k}$$

$$D^j_{\mu\nu}(R) = \sum \frac{n_+!\, n_-!}{\sqrt{n_+'!\, n_-'!\, n_+!\, n_-!}} \frac{1}{k!\,(n_+-k)!\,\ell!\,(n_--\ell)!}$$

$$\langle 00| (a_+)^{n_+'} (a_-)^{n_-'} \times (a_+^\dagger)^k (a_-^\dagger)^{n_+-k} (a_+^\dagger)^\ell (a_-^\dagger)^{n_--\ell} |00\rangle$$

$$R_{++}^k \; R_{+-}^{n_+-k} \; R_{+-}^{\ell} \; R_{--}^{n_--\ell}$$

* this vanishes unless the number of $a_\pm$ $a_\pm^\dagger$ are the same,

* $\langle 0| a^n (a^\dagger)^n |0\rangle = n!$

* $n_+' = k + \ell \implies \ell = n_+' - k$

$n_-' = n_+ - k + n_- - \ell \qquad n_- - \ell = n_- - n_+ + k$

$= n_+ + n_- - k - (n_+' - k)$

$= n_+ + n_- - n_+'$

* this eliminates the ℓ sum

$k : 0 \to n_+ = j + \mu$

$$D^j_{\mu\nu}(R) = \sum \frac{n_+!\, n_-!\, n_+'!\, n_-'!}{\sqrt{n_+!\, n_-!\, n_+'!\, n_-'!}} \frac{1}{k!\,(n_+-k)!\,(n_+'-k)!\,(n_--n_++k)!}$$

$$R_{++}^k \; R_{+-}^{n_+-k} \; R_{+-}^{n_+'-k} \; R_{--}^{n_--n_++k}$$

next express in terms of j, μ, ν

$$\boxed{D^j_{\mu\nu}(R) = \sum_{k=0}^{j+\mu} \frac{\sqrt{(j+\mu)!\,(j-\mu)!\,(j+\nu)!\,(j-\nu)!}}{k!\,(j+\nu-k)\,(j+\mu-\nu)(\nu-\mu+k)!} \; R_{++}^{k}\, R_{+-}^{j+\nu-k}\, R_{+-}^{j+\mu-\nu}\, R_{--}^{k+\nu-\mu}}$$

Remark: This is a homogeneous polynomial in R_{ij} of degree $2j$ with real coefficients

Homework:

Show $D^{\frac{1}{2}}_{\mu\nu}(R) = R_{\mu\nu} \qquad \mu\nu = \pm$

Note θ is the angle of rotation

θ, ϕ can go $0-4\pi$, when we express the rotation in terms of axis and angle.

Adding angular momenta

$$[\vec{J}_a, \vec{J}_b] = 0 \implies$$

$$\vec{J}_{ab} = \vec{J}_a + \vec{J}_b$$

Exercise

$$[J^i_{ab}\; J^j_{ab}] = i \sum_k \varepsilon_{ijk} J^k_{ab}$$

for 2 independent systems basis

$$|j_a m_a\, j_b m_b\rangle$$

$$|j_{ab}\, m_{ab}\rangle$$

There is a unitary change of basis relating these

$$|j_a m_a\, j_b m_b\rangle =$$

$$\sum_{j_{ab}, m_{ab}} |j_{ab}\, m_{ab}\rangle\langle j_{ab}\, m_{ab}|j_a m_a\, j_b m_b\rangle$$

$$|j_{ab}\, m_{ab}\rangle = \sum_{m_a m_b} |j_a m_a\, j_b m_b\rangle\langle j_a m_a\, j_b m_b|j_{ab}\, m_{ab}\rangle$$

x 1

$$J^{+}_{ab}|j_a j_a\, j_b j_b\rangle = (J^{+}_a + J^{+}_b)|j_a j_a\, j_b j_b\rangle = 0$$

$$J^{z}_{ab}|j_a j_a\, j_b j_b\rangle = (j_a + j_b)|j_a j_a\, j_b j_b\rangle$$

$$J^{\mp}_{ab} J^{\pm}_{ab} = J^{2}_{ab} - (J^{z}_{ab})^2 \mp J^{z}_{ab}$$

$$J^{2}_{ab} = (J^{z}_{ab})^2 \pm (J_{ab})^z - J^{\mp}_{ab} J^{\pm}_{ab}$$

$$J^{2}_{ab}|j_a j_a\, j_b j_b\rangle = (j^2_{ab} + j_{ab})|j_a j_a\, j_b j_b\rangle$$

$$= j_{ab}(j_{ab}+1)|j_a j_a\, j_b j_b\rangle$$

This means that $|j_a j_a\, j_b j_b\rangle$ is proportional to $|j_a + j_b\;\; j_a + j_b\rangle$

we define

$$|j_a+j_b\ j_a+j_b\rangle = |j_a j_a\rangle|j_b j_b\rangle$$

which is normalized to 1

$$\langle j_a+j_b\ j_a+j_b|j_a+j_b\ j_a+j_b\rangle =$$

$$\langle j_a j_a|j_a j_a\rangle\langle j_b j_b|j_b j_b\rangle = 1\cdot 1 = 1$$

* remark - this is the only state with eigenvalue of $J^z_{ab} = j_a+j_b$

* $|j_a+j_b\ \mu\rangle$ can be constructed from $|j_a+j_b, j_a+j_b\rangle$ using lowering operators

$$J^-_{ab}|j_a+j_b, j_a+j_b\rangle = (J^-_a + J^-_b)|j_a j_a\ j_b j_b\rangle$$

$$|j_a+j_b, j_a+j_b-1\rangle\sqrt{1(2(j_a+j_b))}$$

$$\sqrt{1\,2j_a}\ |j_a\ j_a-1\ j_b\ j_b\rangle +$$

$$\sqrt{1\cdot 2j_b}\ |j_a\ j_a\ j_b\ j_b-1\rangle$$

$$|j_a+j_b\ j_a+j_b-1\rangle = \sqrt{\frac{j_a}{j_a+j_b}}\ |j_a\ j_a-1, j_b b_b\rangle$$

$$+\sqrt{\frac{j_b}{j_a+j_b}}\ |j_a j_a\ j_b j_b-1\rangle$$

This can be continued by repeated application of $J^-_{ab} = J^-_a + J^-_b$

This gives $2(j_a+j_b)+1$ states

$|j_a+j_b, m\rangle \qquad -(j_a+j_b) \le m \le (j_a+j_b)$

<u>next</u>

There are 2 linearly independent eigenstates of J^2_{ab} with eigenvalue j_a+j_b-1

$|j_a\, j_a-1\rangle|j_b\, j_b\rangle, \quad |j_a\, j_a\, j_b\, j_b-1\rangle$

* $|j_a+j_b, j_a+j_b-1\rangle$ is a linear combination of these states
* The orthogonal complement is also an eigenstate of J^2_{ab} with eigenvalue j_a+j_b-1
* $J^+_{ab}|\text{state}\rangle = 0$ since there is only one state with $J^2_{ab} = j_a+j_b$
* This means it is an eigenstate of J^2_{ab} with eigenvalue $(j_a+j_b-1)(j_a+j_b)$

$$|j_a+j_b-1, j_a+j_b-1\rangle = \left(\sqrt{\frac{j_b}{j_a+j_b}}\,|j_a\, j_{a-1}\, j_b\, j_b\rangle - \sqrt{\frac{j_a}{j_a+j_b}}\,|j_a\, j_a\, j_b\, j_b-1\rangle\right)$$

here c is a phase – we can choose c to be real so the coefficients of the unitary transformation are real

this means that $c = \pm 1$ –
Condon Shortley convention

$\boxed{\text{* coefficient of } |j_> j_> j_< j_< - m\rangle \text{ is positive } (j_> \geq j_<)}$

given $|j_a+j_b-1, j_a+j_b-1\rangle$ all states of the form $|j_a+j_b-1, m\rangle$ can be constructed by applying $J^-_{ab} = J^-_a + J^-_b$

there are 3 independent states with $J^z_{ab} = j_a+j_b-2$

$|j_a j_a \; j_b j_b-2\rangle$
$|j_a j_a-1 \; j_b j_b-1\rangle$
$|j_a j_a-2 \; j_b j_b\rangle$

Two of these are eigenstates of J^2_{ab} with eigenvalues j_a+j_b, j_a+j_b-1 (there are only 2 states with eigenvalues $J^z_{ab} = j_a+j_b-2$

the state $\perp$ to both $|j_a+j_b, j_a+j_b-2\rangle$, $|j_a+j_b-1, j_a+j_b-2\rangle$

This state is proportional to
$|j_a+j_b-2, j_a+j_b-2\rangle$

The sign can be chosen using reality and the Condon Shortley convention.

The states $|j_a+j_b-2, \mu\rangle$ can be constructed using $J^-_{ab} = J^-_a + J^-_b$

This can be repeated until
$j_{ab} = |j_a - j_b|$

Homework: show
$$\sum_{n=|j_a-j_b|}^{j_a+j_b} (2n+1) = (2j_a+1)(2j_b+1)$$

This means this exhausts all states

The coefficients $\langle j_{ab}\mu_{ab} | j_a \mu_a j_b \mu_b\rangle$ are called Clebsch Gordan coefficients. The are constructed to be real

$$C^{j_{ab}\, j_a\, j_b}_{\mu_{ab}\, \mu_a\, \mu_b} \equiv \langle j_{ab}\mu_{ab} | j_a \mu_a j_b \mu_b\rangle$$
$$= \langle j_{ab}\mu_{ab} | j_a \mu_a j_b \mu_b\rangle^*$$