Lecture 39

Identical particles

Consider a system consisting of 2 non interacting identical particles in different states $|a\rangle$ and $|b\rangle$

Physical system

2 identical particles - one in state $|a\rangle$ one in state $|b\rangle$

Hilbert space vector.

$$|ab\rangle_1 = |a\rangle|b\rangle$$

$$P_{ab,ab} = \langle ab|ab\rangle = \langle a|a\rangle\langle b|b\rangle = 1$$

We also have

$$|ab\rangle_2 = |b\rangle|a\rangle$$

This represents the same physical state, but

$${}_1\langle ab | ab \rangle_2 = \langle a|b\rangle\langle b|a\rangle = 0$$

a general linear combination of these identical physical states has

$${}_1\langle ab | (\alpha |ab\rangle_1 + \beta |ab\rangle_2) =$$

$$\alpha \langle a|a\rangle\langle b|b\rangle + \beta \langle a|b\rangle\langle b|a\rangle =$$

$$\alpha$$

where $|\alpha|^2 + |\beta|^2 = 1$

$\therefore$ For identical particles there are a number of distinct vectors (or rays) representing the same physical state

For a system of 3 identical particles we have 3! mutually orthogonal vectors representing the same physical state.

In order to have the standard interpretation we need to have a 1-1 correspondence between physical states and Hilbert space rays

In order to have such a correspondence if $|\psi\rangle$ represents a physical 2 particle state and P_{12} represents an operator that interchanges

particle 1 and 2 then

we must have

$$P_{12}|\psi\rangle = |\psi\rangle e^{i\phi}$$

in order for

$$|\langle\psi|P_{12}|\psi\rangle|^2 = 1$$

this means $e^{i\phi}$ is an

eigenvalue of P_{12}

$$P_{12}^2|\psi\rangle = |\psi\rangle \Rightarrow$$

$$e^{2i\phi} = 1 \qquad \phi = 0, \pi$$

or

$$P_{12}|\psi\rangle = \pm|\psi\rangle$$

when we interchange 2

particles

If we have a three particle system we see that all P_{ij} must have eigenvalues ± 1

Remark

Since P_{12} has real eigenvalues it is a Hermitian operator

$$|\psi\rangle = \Big(\underbrace{\tfrac{1}{2}(1+P_{12})}_{|\psi_+\rangle} + \underbrace{\tfrac{1}{2}(1-P_{12})}_{|\psi_-\rangle}\Big)|\psi\rangle$$

$$P_{12}|\psi_+\rangle = |\psi_+\rangle \qquad P_{12}|\psi_-\rangle = -|\psi_-\rangle$$

Assume $P_{12}|\psi\rangle = |\psi\rangle \quad P_{12}|\phi\rangle = -|\phi\rangle$

$$\langle\psi|\phi\rangle = -\langle\psi|P_{12}|\phi\rangle =$$

$$-\langle\phi|P_{12}|\psi\rangle^{*} = -\langle\phi|\psi\rangle^{*} =$$

$$-\langle\psi|\phi\rangle$$

This means that a physical state is symmetric or antisymmetric under interchange of identical particles

Consider a 3 particle system - assume the systems is symmetric with respect to interchange particles 1 and 2 and 2 and 3 but antisymmetric under change particles 1 and 3.

1 2 3

But $P_{13} = P_{12} P_{23} P_{12}$

2 1 3

2 3 1

which leads to

3 2 1

a contradiction (

identical physical states

represented by ⊥ vectors

This suggests that in order to have a 1-1 correspondence between physical states and vectors they must be represented by states that are either symmetric or antisymmetric under interchange of identical particles

Symmetrization postulate

states of identical integer spin particles must be symmetric under interchange of identical particles states of identical $\frac{1}{2}$ integral spin particles are antisymmetric under interchange of identical particles

Fermions - half integer spin particles

Bosons - integer spin particles

These behave differently because 2 fermions cannot be in the same state, while all bosons can be in the same state.

Permutations

Consider a system of N identical particles and N distinct states. There are $N!$ ways to assign the N particles to the N states; the all represent the same physical state

We can label permutations using

$$\begin{matrix} 1 & 2 & 3 & \cdots & N \\ \sigma(1) & \sigma(2) & \sigma(3) & & \sigma(N) \end{matrix}$$

where for $i \neq j$ $\sigma(i) \neq \sigma(j)$

Example - permutations on 3 objects

$$\begin{matrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 \end{matrix} \equiv \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

we also have

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

we see there are a total of $3! = 6$ permutations

the set of permutations form a group

$$\sigma_1 \qquad\qquad \sigma_2$$

$$\begin{pmatrix} 1 & 2 & 3 \\ \sigma(1) & \sigma(2) & \sigma(3) \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ \sigma(1) & \sigma(2) & \sigma(3) \end{pmatrix}$$

$$(\sigma_2 \cdot \sigma)\, j = \sigma_2(\sigma_1(j))$$

example

$$\sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$(\sigma_2 \sigma_1)(3) = \sigma_2(\sigma_1(3)) = \sigma_2(2) = 2$$

$$(\sigma_2 \sigma_1)(1) = \sigma_2(\sigma_1(1)) = \sigma_2(3) = 1$$

$$(\sigma_2 \sigma_1)(2) = \sigma_2(\sigma_1(2)) = \sigma_2(1) = 3$$

$$\sigma_2 \cdot \sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

* every permutation can be expressed as a product of pairwise transpositions

* the square of every transposition is the identity

* every permutation has an inverse

$$P_\sigma = T_1 T_2 \cdots T_N$$

$$P_\sigma^{-1} = T_N T_{N-1} \cdots T_2 T_1$$

P_N = is the group of permutations on N objects

P_N = has $N!$ elements

<u>Theorem</u>: While there are many ways to express a permutation as a product of transpositions, a given permutation can only be expressed as a product of an odd or even number of transpositions

Proof Assume

$$P_\sigma = T_1 \; T_{2N} = T_1' \; T'_{2M+1}$$

then

$$I = \underbrace{T_{2N} \cdots T_1}_{P_\sigma^{-1}} \underbrace{T_1' \; T'_{2M+1}}_{P_\sigma}$$

This means that the identity can be expressed as the product of an odd number of transposition

If $|A\rangle$ is a vector that is completely antisymmetric with respect to interchanging identical particles

$$|A\rangle = I|A\rangle = P_6^{-1} P_6 |A\rangle = -|A\rangle$$

which is a contradiction

consider

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$\|$$

$$T_{23} T_{12}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad T_{12}$$

$$\begin{pmatrix} 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad T_{23}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad T_{23}$$

$$T_{13} T_{23}$$

$$\begin{pmatrix} 1 & 3 & 2 \\ 2 & 3 & 1 \end{pmatrix} \quad T_{13}$$

$$\therefore \quad \boxed{T_{23} T_{12} = T_{13} T_{23}}$$

also note

$$T_{23} T_{12} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

but

$$T_{12} \; T_{23} \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 & 2 \end{pmatrix}$$

$$T_{12} \; T_{23} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

we see that transpositions performed in different orders generally give different permutations

Definition.

$P(N)$ = set of permutations on N objects for $\sigma \in P(N)$ $|\sigma| = (-)^m$

where m is the number of transpositions needed to make σ

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad |\sigma| = (-)^0 = 1$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = T_{12} \qquad |\sigma| = (-)^1 = -1$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = T_{23} T_{12} \qquad |\sigma| = (-)^2 = 1$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = T_{12} T_{23} \qquad |\sigma| = (-)^2 = 1$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = T_{13} \qquad |\sigma| = (-)^1 = -1$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = T_{23} \qquad |\sigma| = (-)^1 = -1$$

If $|\sigma| = 1$ σ is an even permutation, if $|\sigma| = -1$ σ is an odd permutation

Note If σ is a fixed

and we let $\sigma'' = \sigma\sigma'$

$$\sum_{\sigma'' \in P(n)} f(\sigma'') = \sum_{\sigma'} f(\sigma\sigma')$$

example

$\sigma = T_{12}$

$\sigma' = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$ $\quad \sigma\sigma' = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ $\quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ $\quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ $\quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ $\quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ $\quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

We see the sum runs over the same set of permutations, just in a different order

$$S \equiv \frac{1}{N!} \sum_{\sigma \in \mathcal{P}(N)} P_\sigma$$

$$A \equiv \frac{1}{N!} \sum_{\sigma \in \mathcal{P}(N)} (-1)^{|\sigma|} P_\sigma$$

claim - these are orthogonal projections on symmetric (resp) antisymmetric subspaces of the Hilbert space

We must show

(1) $S^\dagger = S \qquad A^\dagger = A$

(2) $S^2 = S \qquad A^2 = A$

For part 1

$$S^\dagger = \frac{1}{N!} \sum_{\sigma \in \mathcal{P}(N)} P_\sigma^\dagger = \frac{1}{N!} \sum P_\sigma^{-1}$$

(since $T_R = T_R^\dagger$, P is a product of T's, $P^\dagger$ is the product of T's in reverse order, which is P^{-1}

since every permutation has a unique inverse, which is also a permutation and $|\sigma| = |\sigma^{-1}|$ since they both involve the same # of exponent

$$S^{+} = \sum_{\sigma \in P(N)} P(\sigma^{-1}) = \sum_{\sigma^{-1} \in P(N)} P(\sigma) = S$$

similarly

$$A^{+} = \sum_{\sigma \in P(N)} (-)^{|\sigma|} P^{+}(\sigma) =$$

$$\sum_{\sigma \in P(N)} (-)^{|\sigma|} P(\sigma^{-1}) =$$

$$\sum_{\sigma^{-1} \in P(N)} (-)^{|\sigma|} P(\sigma)$$

$$\sum_{\sigma \in P} (-)^{|\sigma|} P(\sigma)$$

We also have

$$S^2 = \frac{1}{N!N!} \sum_{\sigma\sigma'} P(\sigma) P(\sigma')$$

$$= \frac{1}{(N!)^2} \sum_{\sigma} \sum_{\sigma'} P(\sigma\sigma')$$

Let $\sigma'' = \sigma\sigma'$

$$= \frac{1}{(N!)^2} \sum_{\sigma} \left(\sum_{\sigma''} P(\sigma'') \right)$$

$$= S \frac{1}{N!} \sum_{\sigma} = S$$

similarly for A

$$A^2 = \frac{1}{N!N!} \sum_{\sigma\sigma'} (-)^{|\sigma|} P(\sigma) (-)^{|\sigma'|} P(\sigma')$$

$$= \left(\frac{1}{N!}\right)^2 \sum_{\sigma\sigma'} (-)^{|\sigma|+|\sigma'|} P(\sigma\sigma')$$

Let $\sigma'' = \sigma\sigma'$ $\quad |\sigma''| = |\sigma| + |\sigma'|$

$$= \frac{1}{(N!)^2} \sum_{\sigma} \sum_{\sigma'' \in \Theta(N)} (-)^{|\sigma''|} P(\sigma'')$$

$$= \frac{1}{N!} \sum_{\sigma \in \Theta(\ldots)} A = A$$

Finally note that

$$AS|\psi\rangle = \sum_{\sigma\in P(N)} \frac{1}{N!}(-)^{|\sigma|} P(\sigma) S|\psi\rangle$$

$$= \sum_{\sigma\in P(N)} \frac{1}{N!}(-)^{|\sigma|} S|\psi\rangle$$

since $P(\sigma) S = S$ — this vanishes because the number of even and odd permutations are $\frac{N!}{2}$ so they cancel.

Symmetric + Antisymmetric state

$$|\psi_S\rangle = \frac{S|\psi\rangle}{\langle\psi|S|\psi\rangle^{1/2}}$$

$$|\psi_A\rangle = \frac{A|\psi\rangle}{\langle\psi|A|\psi\rangle^{1/2}}$$

If $[P_G\, H] = 0$ for all G

then $[A\, H] = [S\, H] = 0$

an we can find symmetric

or antisymmetric eigenstates.

(only one set is relevant)