Lecture 38

Last time

$$\mathcal{U}(\Lambda\, a) \,|(m\, j)\, p\, \mu\rangle =$$

$$e^{i\Lambda p\cdot a} \sum_{\nu=-j}^{j} |(m\,j)\,\Lambda p\,\nu\rangle\, D^{j}_{\nu\mu}(R_{\omega}(\Lambda p)) \times \sqrt{\frac{\omega(\Lambda p)}{\omega(p)}}$$

where

$$R_{\omega}(\Lambda p) = B^{-1}(\Lambda p)\,\Lambda\, B(p)$$

$$\omega(p) = \sqrt{m^2c^2 + \bar{p}^2}$$

We showed that $D^{j}_{\mu\nu}(A)$ satisfied

$$D^{j}(A')\,D^{j}(A) = D^{j}(A'A)$$

We could also combine spins with ordinary Clebsch Gordon coefficients we define

$$|(m j)\,\bar{p}\,\mu\rangle_{cov} \equiv$$

$$\sum |(m j)\,\bar{p}\,\nu\rangle \sqrt{\omega(p)}\; D^{j}_{\nu\mu}(B^{-1}(p))$$

and

$$|(m j)\,\bar{p}\,\mu\rangle_{\overline{cov}} \equiv$$

$$\sum |(m j)\,\bar{p}\,\nu\rangle \sqrt{\omega(p)}\; D^{j}_{\nu\mu}(B^{\dagger}(p))$$

these transform like

$$\mathcal{U}(\Lambda\, 0)\,|(m j)\,\bar{p}\,\mu\rangle_{cov} =$$

$$\sum_{\nu} |(m j)\,\overline{\Lambda p},\nu\rangle\, D^{j}_{\nu\mu}(\Lambda)$$

$$\mathcal{U}(\Lambda\, 0)\,|(m j)\,\bar{p}\,\mu\rangle_{\overline{cov}}$$

$$\sum_{\nu} |(m j)\,\overline{\Lambda p}\,\nu\rangle\, D^{j}_{\nu\mu}(\Lambda^{\dagger\,-1})$$

These transformations are identical for rotations but not for general Lorentz transformations

We also showed

$$I = \int d^4p \; |(mj)\,p\,\mu\rangle_{cov} \; \delta(p^2+m^2)\,\Theta(p^0)\, D^{j}_{\mu\nu}(p\cdot\sigma) \; {}_{cov}\langle(mj)\,p\,\nu|$$

$$= \int d^4p \; |(mj)\,p\,\mu\rangle_{\overline{cov}} \; \delta(p^2+m^2)\,\Theta(p^0)\, D^{j}_{\mu\nu}(\Pi p\cdot\sigma) \; {}_{\overline{cov}}\langle(mj)\,p\,\nu|$$

these equations mean

$$\langle\Psi|a\rangle = \int \langle\Psi|(mj)\,\bar{p}\mu\rangle_{cov}$$

$$\delta(p^2 - m^2)\,\theta(p_0)\,D^{j}_{\mu\nu}(p.\sigma)$$

$$_{cov}\langle(mj)\,p\nu|\Phi\rangle$$

or

$$= \int \langle\Psi|(mj\,p\bar{\mu}\rangle_{\overline{cov}}$$

$$\delta(p^2 - m^2)\,\theta(p^0)\,D^{j}_{\mu\nu}(\Pi p.\sigma)$$

$$_{\overline{cov}}\langle(mj)\,\bar{p}\nu|\Phi\rangle$$

the cov representation is called the right handed representation while the $\overline{cov}$ is called the left handed representation recall

$$(A^T)^{-1} = \sigma_2 A^* \sigma_2$$

$$\sigma_2 X^* \sigma_2 = \text{space reflected } X$$

$$X' = A X A^+$$

$$\tilde{X} = \sigma_2 X^* \sigma_2 \qquad \tilde{A} = (A^+)^{-1}$$

$$\tilde{X} = \tilde{A} X \tilde{A}^+$$

define <u>inequivalent</u> representations of $SL(2C)$

(meaning there is <u>no</u> fixed matrix $C$ satisfying

$$C \tilde{A} C^{-1} = A \qquad )$$

$X$ and $\tilde{X}$ are related by space reflection this means that if we space reflect a right handed state, it will not transform correctly

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In order to use covariant representations where space reflection symmetry is important we take a direct sum of a right and left handed spinor.

$$u_{\uparrow}(0)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$$

spin ↑ right

spin ↑ left

$$u(0)_{\downarrow} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$ right down

left down

then we define

$$u(p) = \begin{pmatrix} D^{\frac{1}{2}}(B(p)) & 0 \\ 0 & D^{\frac{1}{2}}(B^{-1}(p)) \end{pmatrix} \begin{pmatrix} u_{\uparrow\downarrow}(0) \end{pmatrix}$$

for spin $\frac{1}{2}$

$$D^{\frac{1}{2}}(B(p)) = B(p) =$$

$$\cosh\left(\frac{\rho}{2}\right) + \hat{p}\cdot\bar{\sigma} \sinh\left(\frac{\rho}{2}\right) =$$

$$\sqrt{\frac{\cosh \rho + 1}{2}} + \hat{p}\cdot\bar{\sigma}\sqrt{\frac{\cosh \rho - 1}{2}} =$$

$$\sqrt{\frac{p^0/mc + 1}{2}} + \hat{p}\cdot\bar{\sigma}\sqrt{\frac{p^0/mc - 1}{2}} =$$

$$\sqrt{\frac{p^0 + mc}{2mc}} + \hat{p}\cdot\bar{\sigma}\sqrt{\frac{p^0 - mc}{2mc}} =$$

$$\frac{1}{\sqrt{2mc(p^0 + mc)}} (p^0 + mc + \bar{p}\cdot\bar{\sigma})$$

while $B(\Pi p) =$

$$\frac{1}{\sqrt{2mc(p^0+mc)}}\left(p^0+mc-\bar{p}\cdot\bar{\sigma}\right)$$

$$S(B(p)) = \begin{pmatrix} B(p) & 0 \\ 0 & B(\Pi p) \end{pmatrix} =$$

$$\frac{1}{\sqrt{2mc(p^0+mc)}}\begin{pmatrix} p^0+mc+p^3 & p^1-ip^2 & 0 & 0 \\ p^1+ip^2 & p^0+mc-p^3 & 0 & 0 \\ 0 & 0 & p^0+mc-p^3 & -p^1+ip^2 \\ 0 & 0 & -p^1-ip^2 & p^0+mc+p^3 \end{pmatrix}$$

$$u_\uparrow(p) = \frac{1}{2\sqrt{mc(p^0+mc)}}\begin{pmatrix} p^0+mc+p^3 \\ p^1+ip^2 \\ p^0+mc-p^3 \\ -p^1-ip^2 \end{pmatrix}$$

$$u_\downarrow(p) = \frac{1}{2\sqrt{mc(p^0+mc)}}\begin{pmatrix} p^1-ip^2 \\ p^0+mc-p^3 \\ -p^1+ip^2 \\ p^0+mc+p^3 \end{pmatrix}$$

To proceed we define

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \sigma_2 \sigma_\mu^* \sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_\mu \\ \pi \sigma_\mu & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \sigma_\mu \\ \tilde{\sigma}_\mu & 0 \end{pmatrix}$$

we also define

$$\gamma^\mu = \sum \eta^{\mu\nu} \gamma_\mu$$

which changes the sign of $\gamma_0$

These matrices have 2 important properties

① $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu =$

$$\begin{pmatrix} 0 & \sigma_\mu \\ \tilde{\sigma}_\mu & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_\nu \\ \tilde{\sigma}_\nu & 0 \end{pmatrix} + \quad \nu \leftrightarrow \mu$$

$$\begin{pmatrix} \sigma_\mu \tilde{\sigma}_\nu & 0 \\ 0 & \tilde{\sigma}_\mu \sigma_\nu \end{pmatrix} + (\nu \leftrightarrow \mu) =$$

$= -2\eta_{\mu\nu}$ (complete in HW)

The other important property is

$$S(A)\gamma_\mu S(A)^{-1} =$$

$$\begin{pmatrix} A & 0 \\ 0 & (A^\dagger)^{-1} \end{pmatrix} \begin{pmatrix} 0 & \sigma_\mu \\ \tilde{\sigma}_\mu & 0 \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & A^\dagger \end{pmatrix}$$

$$\begin{pmatrix} A & 0 \\ 0 & (A^\dagger)^{-1} \end{pmatrix} \begin{pmatrix} 0 & \sigma_\mu A^\dagger \\ \tilde{\sigma}_\mu A^{-1} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & A\sigma_\mu A^\dagger \\ (A^\dagger)^{-1}\tilde{\sigma}_\mu A^{-1} & 0 \end{pmatrix} =$$

$$\sum_\nu \begin{pmatrix} 0 & \sigma_\nu \\ \tilde{\sigma}_\nu & 0 \end{pmatrix} \Lambda^\nu{}_\mu$$

$$\sum \gamma_\nu \Lambda^\nu{}_\mu$$

which we write as

$$\boxed{S(\Lambda)\gamma_\mu S(\Lambda)^{-1} = \sum \gamma_\nu \Lambda^\nu{}_\mu}$$

Next consider

$$\left(\sum p^\mu \gamma_\mu - mcI\right) u(p) =$$

$$\left(\sum p^\mu \gamma_\mu - mcI\right) S(B(p))\, u(0) =$$

$$S(B(p))\, S^{-1}(B(p)) \left(\sum p^\mu \gamma_\mu - mcI\right) S(B(p))\, u(0) =$$

$$S(B(p)) \left(\sum p^\mu \left(\gamma_\nu B^{-1}(p)^\nu{}_\mu\right) - mcI\right) u(0)$$

$$S(B(p)) \left(\sum \gamma_\nu B^{-1}(p)^\nu{}_\mu p^\mu - mcI\right) u(0)$$

$$S(B(p)) \left(\gamma_0 mc - mcI\right) u(0)$$

$$S(B(p)) \begin{pmatrix} -mcI & mcI \\ mcI & -mcI \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = 0$$

if we multiply by a plane wave each

$$u(p)\, e^{i p\cdot x/\hbar}$$

with $p_0 = \sqrt{m^2c^2 + \bar{p}^2}$

$$0 = \left(\sum \gamma_\mu p^\mu - mc\right) u(p) = 0 \Rightarrow$$

$$* \quad 0 = \int \left(\sum \gamma_\mu p^\mu - mc\right) u(p)\, e^{i p\cdot x/\hbar}\, d^3p = 0$$

$$i p\cdot x/\hbar = -i p^0 x^0/\hbar + i \bar{p}\cdot\bar{x}/\hbar$$

$$\frac{\partial}{\partial x_0} = -\frac{i p^0}{\hbar} \qquad \frac{\partial}{\partial x_i} = i\frac{p_i}{\hbar}$$

$$p^0 = i\hbar\frac{\partial}{\partial x_0} \qquad \bar{p} = -i\hbar\bar{\nabla}$$

using these in the equation

$$i\hbar\left(\gamma_0\frac{\partial}{\partial x_0} - \sum_i \gamma_i\frac{\partial}{\partial x_i} + mc\right)\psi(x) = 0$$

$$\psi(x) = \int u(p)\, e^{i p\cdot x/\hbar}\, d^3p$$

This equation is called the Dirac equation

* There is one solution for spin ½ in the rest frame
* The equation holds for all frames
* These are both positive energy solutions
* Coupling to E+M

$$\sum \left( \gamma_\mu \left(p^\mu - \tfrac{e}{c} A^\mu\right) - mc \right) \psi(x) = 0$$

give the almost corrected magnetic moment

Remark - in addition

to

$$u_{\uparrow}(0) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$$

$$u_{\downarrow}(0) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

we can also consider

$$V_{\uparrow}(0) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}$$

$$V_{\downarrow}(0) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}$$

$$V(p) = \mathcal{U}(B(p))\,V(0)$$

these objects also transform

covariantly - these solutions

require that $p^{\mu} = (-m, 0, 0, 0)$

The fix

The Dirac equation is a classical field equation that must be quantized like Maxwells equations.