Lecture 37

Last time we constructed $\mathcal{U}(\Lambda, a)$ for a free particle of mass m and spin $\frac{1}{2}$ (j)

$$\mathcal{U}(\Lambda,a)\,|(mj)\,\bar{p}\,\mu\rangle =$$

$$\sum_{\nu=-j}^{j} e^{i\Lambda p\cdot a}\,|(mj)\,\overrightarrow{\Lambda_p}\,\nu\rangle\, D^{j}_{\nu\mu}\left(B^{-1}(\Lambda p)\Lambda B(p)\right)\times \sqrt{\frac{\omega(\Lambda p)}{\omega(p)}}$$

where $\omega(p) = \sqrt{\bar{p}^2 + m^2c^2}$

(these are called positive mass - positive energy irreducible unitary representations of the Poincare group consider

$$\frac{d}{dt}\langle (mj)\,\bar{p}\mu \,|\, \Psi(t)\rangle =$$

$$\frac{d}{dt}\langle (mj)\,\bar{p}\mu \,|\, e^{-iHt/\hbar} \,|\, \Psi(0)\rangle =$$

$$-\frac{i}{\hbar}\sqrt{m^2c^4+\bar{p}^2c^2}\,\langle (mj)\,\bar{p}\mu \,|\, \Psi(t)\rangle$$

or

$$i\hbar\frac{d}{d(ct)}\langle (mj)\,\bar{p}\mu \,|\, \Psi(t)\rangle =$$

$$\sqrt{m^2c^2+\bar{p}^2}\,\langle (mj)\,\bar{p}\mu \,|\, \Psi(t)\rangle$$

$$\boxed{\begin{aligned} &i\hbar\frac{d}{dx^0}\langle (mj)\,\bar{p}\mu \,|\, \Psi(t)\rangle = \\ &\quad\sqrt{m^2c^2+\bar{p}^2}\,\langle (mj)\,\bar{p}\mu \,|\, \Psi(t)\rangle \end{aligned}}$$

This is called the relativistic Schrödinger equation

It is a correct equation but it was discarded because making the replacement $p^\mu \to p^\mu - \frac{e}{c} A^\mu$ did not lead to the correct electron magnetic moment.

In addition it did not result in a covariant treatment of all four components of p^μ

* Applying the time derivative a second time gives

$$-\hbar^2 \frac{\partial^2}{\partial x^{0\,2}} \langle (m\,j)\, p\mu \,|\, \Psi(t) \rangle =$$

$$(m^2c^2 + \vec{p}^{\,2}) \langle (m\,j)\, p\mu \,|\, \Psi(t) \rangle$$

using $\bar{p} = -i\hbar \bar{\nabla}_x$ gives

$$\boxed{\hbar^2 \left(\frac{\partial^2}{\partial x^{02}} - \nabla^2 + m^2 c^2 \right) \langle (m s) | \bar{p} u | \Psi(t) \rangle = 0}$$

This equation is called the Klein Gordon Schrodinger equation.

① Introducing the second derivative means that it also has negative energy solutions

$$\langle (m s) \bar{p} u | \Psi(t) \rangle e^{i E t / \hbar}$$

(these are unstable since the negative energies are not bounded from below

(2) Probabilities are not conserved

$$i\hbar \frac{d}{dt} \langle \psi(t) | \varphi(t) \rangle =$$

$$\langle \psi(t) | H - H^{\dagger} | \varphi(t) \rangle$$

if one state has positive energy and one has negative energy the 2 terms add rather than subtract -

probability conservation requires a first order equation

* Dirac looked for an equation that has first derivatives in time in order to

conserve probability and also first order in the space derivatives to have an equation that transforms covariantly under Lorentz transformations

→ I will leave the original derivation for homework and give an alternate derivation

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Recall

① $D^{j}_{\mu\nu}(R)$ is a polynomial of degree $2j+1$ in the components R_{ij} of $R \in SU(2)$

(2) $R \in SU(2) \Rightarrow$

$$R = \cos\left(\frac{\theta}{2}\right) I - i \bar{\sigma} \cdot \hat{\theta} \sin\left(\frac{\theta}{2}\right)$$

$\Rightarrow$ R is an entire function of angles

$\Rightarrow D^{j}_{\mu\nu}(R)$ is an entire (polynomial) function of components of R

$\therefore D^{j}_{\mu\nu}(R(\bar{\theta}))$ is an entire function of $\bar{\theta}$

$$\Rightarrow \boxed{\begin{aligned} 0 &= \sum D^{j}_{\mu\alpha}(R_2) D^{j}_{\alpha\nu}(R_1) - D^{j}_{\mu\nu}(R_2 R_1) \\ 0 &= D^{j_1}_{\mu_1\nu_1}(R) D^{j_2}_{\mu_2\nu_1}(R) - \sum_{\mu\nu j} \langle j_1 \mu_1 j_2 \mu_1 | j\mu \rangle D^{j}_{\mu\nu}(R) \langle j_1 \nu_1 j_2 \mu_1 | j\nu \rangle \end{aligned}}$$

It follows that these equations also hold when $\bar{\theta} \to \bar{z} = \bar{\theta} - i\bar{p}$, so they also hold for $R \in SU(2) \to A$ in $SL(2\mathbb{C})$

This means that we can rewrite

$$\mathcal{U}(\Lambda, 0)\,|(mj)\,p\mu\rangle =$$

$$\sum |(mj)\,\Lambda p\,\nu\rangle \sqrt{\frac{\omega(\Lambda p)}{\omega(p)}}\; D^{j}_{\nu\mu}\left(B^{-1}(\Lambda p)\,\Lambda\, B(p)\right)$$

as

$$\sum_{\nu} \mathcal{U}(\Lambda\, 0)\,|(mj)\,\bar{p}\nu\rangle \sqrt{\omega(p)}\; D^{j}_{\nu\mu}\left(B^{-1}(p)\right) =$$

$$\sum_{\nu\alpha} |(mj)\,\Lambda p\,\alpha\rangle \sqrt{\omega(\Lambda p)}\; D^{j}_{\alpha\nu}\left(B^{-1}(\Lambda p)\right) D_{\nu\mu}(\Lambda)$$

this suggests that we define a new kind of plane wave state

$$|(mj)\,\bar{p}\,\mu\rangle_{cov} \equiv \sum_{\nu=-j}^{j} |(mj)\,\bar{p}\,\nu\rangle \sqrt{\omega(p)}\, D^{j}_{\nu\mu}(\bar{B}^{-1}(p))$$

with this definition the equation on the previous page becomes

$$\boxed{\mathcal{U}(\Lambda a)\,|(mj)\,\bar{p}\,\mu\rangle_{cov} = \sum_{\nu=-j}^{+j} e^{i\Lambda p\cdot a}\, |(mj)\,\overline{\Lambda p}\,\nu\rangle_{cov}\, D^{j}_{\nu\mu}(\Lambda)}$$

In this case the spins transform under a $2j+1$ dimensional representation of $SL(2\mathbb{C})$.

with these covariant states we can write the identity as

$$I = \sum_{\mu=-j}^{j} \int |(mj)\bar{p}\mu\rangle\, d^3p\, \langle (mj)\bar{p}\mu| =$$

$$\sum_{\alpha\,\mu\nu=-j}^{j} \int |(mj)\bar{p}\mu\rangle_{cov} \frac{D^{j}_{\mu\alpha}(B(p))\, D^{j}_{\alpha\nu}(B^{+}(p))}{\omega(p)}$$

$$d^3p\ {}_{cov}\langle (mj)\bar{p}\mu|$$

since $B^{+}(p) = B(p)$ that

$$\sum_{\alpha} D^{j}_{\mu\alpha} B(p))\, D^{j}_{\alpha\nu}(B^{+}(p)) = D^{j}_{\mu\nu}(B^{2}(p))$$

but

$$B^{2}(p) = e^{\bar{p}\cdot\bar{\sigma}} = \cosh\rho\, I + \hat{p}\cdot\bar{\sigma}\sinh\rho$$

$$= \frac{p^{0}}{mc} I + \frac{\bar{p}}{mc}\cdot\bar{\sigma} = \frac{p}{mc}\cdot\sigma$$

so this becomes

$$I = \sum_{\mu\nu} \int \frac{d^3\bar{p}}{\omega(p)} |(mj)\bar{p}\mu\rangle_{cov} D^j_{\mu\nu}(p\cdot\sigma) \times$$

$$_{cov}\langle (mj)\,p\,\nu|$$

finally recall

$$\int \frac{d^3p}{\omega(p)} = \int d^4p \, 2\,\delta(p^2+m^2c^2)\,\theta(p^0)$$

$$I = \int d^4p \, |(mj)\bar{p}\mu\rangle_{cov} \cdot 2\delta(p^2+m^2c^2)\,\theta(p^0)$$

$$D^j_{\mu\nu}(p\cdot\sigma)\,_{cov}\langle (mj)\,\bar{p}\,\mu|$$

* Note that in SU(2)
unitarity implies

$$R^\dagger = R^{-1}$$

or equivalently

$$R = (R^\dagger)^{-1}$$

while this identity holds in SU(2) it does not hold in SL(2,C)

$$(A^\dagger)^{-1} = (e^{\vec{z}\cdot\vec{\sigma}})^{\dagger -1} = (e^{-\vec{z}\cdot\vec{\sigma}})^{\dagger}$$

$$= e^{-z^*\cdot\sigma} \neq A$$

in addition there is no similarity transformation relating A and $(A^\dagger)^{-1}$.

since

$$e^{-z^*\cdot\sigma} = e^{z^*\sigma_2\bar{\sigma}^*\sigma_2} = \sigma_2 e^{\bar{z}^*\cdot\bar{\sigma}^*}\sigma_2$$

$$= \sigma_2 A^* \sigma_2 = (A^\dagger)^{-1}$$

we note that

$$\tilde{X} = \sigma_2 X^* \sigma_2$$

$$\det \tilde{X} = \det \sigma_2 X^* \sigma_2 =$$

$$\det \sigma_2 \det X^* \det \sigma_2 =$$

$$(\det \sigma_2)(\det X)^*(\det \sigma_1) =$$

$$1 \cdot (x^{0^2} - \bar{x}^{1})^* \cdot 1 = x^{0^2} - \bar{x}^2$$

$$\sigma_2 X^* \sigma_2 = \begin{pmatrix} x^0 & -x^1+ix^2 \\ -x^1-ix^2 & x^0 \end{pmatrix}$$

this has the same form as X with the signs of the space components reversed — this corresponds to a space reflection

8. while ordinary covariant spinors transform under SL(2C) — space reflected spinors transform under $(SL(2C))^{*T}$.

Replacing R by $(R^{+})^{-1}$ in the wigner rotatn

$$(R_w^{+})^{-1} = \left(B^{-1}(\Lambda p)\,\Lambda\, B(p) \right)^{+\,-1} =$$

$$= B^{+}(\Lambda p)(\Lambda^{+})^{-1} B^{-1}(p)$$

$$= B(\Lambda p)(\Lambda^{+})^{-1} B^{-1}(p)$$

$$= B^{-1}(\Pi\Lambda p)(\Lambda^{+})^{-1} B(\Pi p)$$

where Π is the space reflectn operatn

$$|(mj)\,p\mu\rangle_{\overline{cov}} =$$

$$\sum_{\nu} |(mj)\,p\nu\rangle\, \omega(p)\, D_{\nu\mu}(B^{+}(p)) =$$

$$\sum_{\nu} |(mj)\,p\nu\rangle\, \omega(p)\, D_{\nu\mu}(B(p))$$

and

$$I = \sum_{\mu\nu\alpha} \int |(mj)\, p\mu\rangle_{\overline{cov}} \frac{D^j_{\mu\alpha}(B^{-1}(\pi))\, D^j_{\alpha\nu}(B^{-1}(p))}{\omega(m)}\, d^3p$$

$$\times \langle (mj)\, p\mu \rangle_{c\cdot\bar{v}}$$

$$\boxed{I = \int |(mj)\, p\mu\rangle_{\overline{cov}}\, D^j_{\mu\nu}(\pi p\cdot\sigma) \times d^4p \quad 2\,\delta(p^2+m^2)\,\Theta(p_0)\, \langle (mj)\, p\mu |_{\overline{cov}}}$$

while it is possible to transfer between the Lorentz and Poincare covariant representations, it is not possible to implement space reflection consistently in the covariant representations

Dirac equation - Dirac spinors

one way to treat space reflection covariantly is to use the direct sum of a Lorentz covariant representation and a reflected Lorentz covariant representation (these are called $\boxed{\text{right}}$ and $\boxed{\text{left}}$ handed representations)

$$u(0\uparrow) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$$

$$u(0\downarrow) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

we define

$$u(p) = \begin{pmatrix} B(P/m) & 0 \\ 0 & B(\Pi p/m) \end{pmatrix} u(0, u)$$

recall

$$B(P/m) = \cosh\left(\frac{\rho}{2}\right) + \hat{p}\cdot\bar{\sigma}\sinh\left(\frac{\rho}{2}\right) =$$

$$= \sqrt{\frac{\cosh\rho + 1}{2}} + \hat{p}\cdot\bar{\sigma}\sqrt{\frac{\cosh\rho - 1}{2}}$$

$$= \sqrt{\frac{p^0 + mc}{2mc}} + \hat{p}\cdot\bar{\sigma}\sqrt{\frac{p^0 - mc}{2mc}}$$

$$= \frac{1}{\sqrt{2mc(p^0 + mc)}} \cdot ((p^0 + mc)I + \bar{p}\cdot\bar{\sigma})$$

similarly

$$B(\pi p/m) =$$

$$\frac{1}{\sqrt{2mc(p^0 + mc)}}((p^0 + mc)I - \bar{p}\cdot\bar{\sigma})$$

so

$$u_1(p) = \frac{1}{\sqrt{2mc(p^0+mc)}} \begin{pmatrix} p^0+mc+p^3 \\ p^1+ip^2 \\ p^0+mc-p^3 \\ -p^1-ip^2 \end{pmatrix}$$

$$u_2(p) = \frac{1}{\sqrt{2mc(p^0+mc)}} \begin{pmatrix} p^1-ip^2 \\ p^0+mc-p^3 \\ -p^1+ip^2 \\ p^0+mc+p^3 \end{pmatrix}$$

Next we define some operators that act on the 4 component spinor space

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \Pi\sigma_\mu & 0 \end{pmatrix} \qquad \Pi\sigma_\mu = (I, -\vec{\sigma})$$

$$\gamma^\mu = \begin{pmatrix} 0 & -\Pi\sigma_\mu \\ -\sigma_\mu & 0 \end{pmatrix}$$

The matrices are one representation of matrices called Gamma matrices (there are several representations related by similarity transformations)

They satisfy

(1) $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2\eta^{\mu\nu} = -2\eta_{\mu\nu}$

(Homework)

(2) for $S(\Lambda) = \begin{pmatrix} \Lambda & 0 \\ 0 & (\Lambda^\dagger)^{-1} \end{pmatrix}$

$S(\Lambda)\gamma_\mu S^{-1}(\Lambda) =$

$\begin{pmatrix} \Lambda & 0 \\ 0 & (\Lambda^\dagger)^{-1} \end{pmatrix} \begin{pmatrix} 0 & \sigma_\mu \\ \Pi\sigma_\mu & 0 \end{pmatrix} \begin{pmatrix} \Lambda^{-1} & 0 \\ 0 & \Lambda^\dagger \end{pmatrix} =$

$\begin{pmatrix} \Lambda & 0 \\ 0 & (\Lambda^\dagger)^{-1} \end{pmatrix} \begin{pmatrix} 0 & \sigma_\mu \Lambda^\dagger \\ \Pi\sigma_\mu \Lambda^{-1} & 0 \end{pmatrix} =$

$\begin{pmatrix} 0 & \Lambda \sigma_\mu \Lambda^\dagger \\ (\Lambda^\dagger)^{-1}\Pi\sigma \Lambda^{-1} & 0 \end{pmatrix} = \sum_\nu \gamma_\nu \Lambda^\nu{}_\mu$

It follows that

$$\left(\sum_\mu p^\mu \gamma_\mu - mc\right) u(p) =$$

$$\left(\sum_\mu p^\mu \gamma_\mu - mc\right) S(B(p))\, u(0) =$$

$$S(B(p)) \left[\sum p^\mu S^{-1}(B(p))\, \gamma_\mu\, S(B(p)) - mc \right] u(0)$$

$$S(B(p)) \left[\sum_{\mu\nu} p^\mu \gamma_\nu B^{-1}(p)^\nu{}_\mu - mc \right] u(0)$$

$$S(B(p)) \left[B^{-1}(p)^\nu{}_\mu p^\mu \gamma_\nu - mc \right] u(0)$$

$$S(B(p)) \left[mc\gamma_0 - mc \right] u(0)$$

$$S(B(p))\, mc \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} u(0)$$

which gives 0 for $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

If we consider

$$u(p)\, e^{i p \cdot x/\hbar}$$

this is a solution to

$$\sum_\mu (\gamma_\mu p^\mu - mc)\, u(p)\, e^{ip\cdot x/\hbar}$$

$$\sum_\mu (\gamma^\mu p_\mu - mc)\, u(p)\, e^{ip\cdot x/\hbar}$$

$$\sum_\mu \left(-i\hbar \gamma^\mu \frac{\partial}{\partial x^\mu} - mc\right) u(p)\, e^{ip\cdot x} = 0$$

This equation is called the Dirac equation. It has the desirable features that

① It is frame independent (multiply by $S(\Lambda)$)

② It involves only first derivative in space + time

③ $\sum_\mu \left(\gamma_\mu (p^\mu - \frac{e}{c}A^\mu) - mc\right) u(p)\, e^{ip\cdot x} = 0$

gives the correct electron magnetic moment (up to small corrections) we have just constructed

2 solutions - If we let

$$v(0)_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \quad v_{+2}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

$$v(p) = S(B(p))\, v(0)$$

$$v(p)\, e^{-ip\cdot x}$$

results in negative

energy solutions of the

Dirac equation - so it

does not fix the negative

energy problem

Dirac assumed that

since the electrons are

fermions the negative

energy states are all

occupied

the absence of a negative energy electron behaves like a positively charged electron in this picture.

Dirac used this to predict the existence of an antielectron.

the problem with this interpretation is that the anti Hydrogen atom - bound state of a negative energy proton and electron is a negative energy boson which is not protected from decay.

the correct interpretation of the Dirac equation is that it is a classical field equation that need to be quantized like Maxwell's equations