Lecture 36

Recall

$|(m\,\tfrac{1}{2})\,\bar{p}\,\mu\rangle$

We used

① $\mathcal{U}(R,0)\,|(m\,\tfrac{1}{2})\,\bar{0}\,\mu\rangle =$

$$\sum_{\nu} |(m\,\tfrac{1}{2})\,\bar{0}\,\nu\rangle\, D^{\frac{1}{2}}_{\nu\mu}(R)$$

② $\mathcal{U}(B(p/m),0)\,|(m\,\tfrac{1}{2})\,\bar{0}\,\mu\rangle =$

$$|(m\,\tfrac{1}{2})\,\bar{p}\,\mu\rangle \sqrt{\frac{\omega_m(p)}{m}}$$

where $\omega_m(p) = \sqrt{p^2 + mc^2}$

③ $\mathcal{U}(I,a)\,|(m\,\tfrac{1}{2})\,0\,\mu\rangle =$

$$|(m\,\tfrac{1}{2})\,0\,\mu\rangle\, e^{-i a^0 mc}$$

These are all unitary and can be combined to give

$$\boxed{\begin{aligned}&\mathcal{U}(\Lambda\, a)\,|(m\tfrac{1}{2})\,\bar{p}\,\mu\rangle = \\ &\sum_{\nu=-1/2}^{1/2} |(m\tfrac{1}{2})\,\overrightarrow{\Lambda p},\nu\rangle\, D^{\frac{1}{2}}_{\nu\mu}\left(B^{-1}(\Lambda p)\,\Lambda\, B(p)\right) \times \\ &\qquad\qquad \sqrt{\frac{\omega_m(\Lambda p)}{\omega_m(p)}}\; e^{i\Lambda p\cdot a}\end{aligned}}$$

This gives a unitary representation of the Poincaré group on the single electron hilbert space

$$i\hbar \frac{d}{dt} \langle (m\tfrac{1}{2})\,\bar{p}\,\mu|\,\mathcal{U}(I,t)\,|\Psi\rangle$$

$$c\sqrt{p^2+m^2c^2}\,\langle (m\tfrac{1}{2})\,\bar{p}\,\mu|\,\mathcal{U}(I,t)\,|\Psi\rangle$$

which the the expected generalization of the schrodinger equation

This equation was already derived by Schrodinger, but it was discarded because $\bar{p} \to \bar{p} - \frac{e}{c}\bar{A}$ did not give the experimental electron magnetic moment.

Also, there were objections because it did not seem to have nice transformation properties under Lorentz transformations

This could be made manifestly covariant by applying $i\hbar \frac{\partial}{\partial t}$ twice

$$-\hbar^2 \frac{\partial^2}{\partial t^2} \langle (m \tfrac{1}{2}) \bar{p} \mu | U(I t) | \psi \rangle =$$

$$(c^2 p^2 + m^2 c^4) \langle (m \tfrac{1}{2}) \bar{p} \mu | U(I t) | \psi \rangle$$

or

$$0 = \left(\hbar^2 \frac{\partial^2}{\partial x^{0\,2}} - \hbar^2 \frac{\partial^2}{\partial \bar{x}^2} + m^2 c^2 \right) \langle (m \tfrac{1}{2}) p \mu | U(t) | \psi \rangle$$

This is called the Klein Gordon equation. While it treats the space and time derivatives symmetrically, it has 2 problems

① The derivative introduces a second unphysical negative energy solution

② quantum probabilities

are not conserved

in time

$$i\hbar \frac{\partial}{\partial t} \Psi = H\Psi \Rightarrow$$

$$\frac{\partial}{\partial t} \langle \Psi | \varphi \rangle = i\hbar \langle \Psi | (H^{+} - H) | \varphi \rangle$$

vanishes for $H = H^{+}$, this

does not when $|\Psi\rangle$ is

a negative energy solution

and $|\varphi\rangle$ is a positive

energy solution

Dirac wanted a covariant

equation that was linear

in the space + time derivatives

Rather than follow Dirac's

path I derive the

solution using Wigner.

representation

Recall

$$R = e^{\frac{i}{2}\bar{\Phi}\cdot\sigma}$$

$D^{j}_{\mu\nu}(R)$ = homogeneous polynomial in components of R

$\therefore$ $D(R)$ is an entire function of $\bar{\Phi}$. This means that the identity

$$\left(\sum_{\nu} D^{j}_{\mu\nu}(R_2)\, D^{j}_{\nu\alpha}(R_1) - D^{j}_{\mu\alpha}(R_2 R_1)\right) = 0$$

$$D^{j}_{\mu\nu}(R) - \langle j\mu \,|\, j_1\mu_1 j_2\mu_2\rangle \times D^{j_1}_{\mu_1\nu_1}(R)\, D^{j_2}_{\mu_2\nu_2}(R) \times \langle j_1\nu_1 j_2\nu_2 \,|\, j\nu\rangle = 0$$

hold for complex angles
- ie replace R by a $SL(2\mathbb{C})$ matrix

This means we can write

$$\mathcal{U}(\Lambda,0)\,|(m\,j)\,p\,\mu\rangle =$$

$$\sum |(m\,j)\,\Lambda p\,\nu\rangle\, D^{j}_{\nu\mu}\left(B^{-1}(\Lambda p)\,\Lambda\, B(p)\right)\sqrt{\frac{\omega(\Lambda p/m)}{\omega(p/m)}} =$$

multiply on the right by

$D(B^{-1}(p)) \times \sqrt{\omega(p/m)}$ gives

$$\boxed{\begin{aligned}&\sum_{\nu} \mathcal{U}(\Lambda\,0)\,|(m\,j)\,p\,\nu\rangle\, D^{j}_{\nu\mu}(B^{-1}(p))\sqrt{\omega_m(p)} = \\ &\sum |(m\,j)\,\Lambda p\,\nu\rangle\, D^{j}_{\nu\alpha}(B^{-1}(\Lambda p))\sqrt{\omega_m(\Lambda p)}\; D^{j}_{\alpha\mu}(\Lambda)\end{aligned}}$$

This suggests defining a covariant wave functions

$$\boxed{|(m\,j)\,p\,\nu\rangle_{cov} \equiv \sum_{\alpha} |(m\,j)\,p\,\alpha\rangle\, D^{j}_{\alpha\nu}(B^{-1}(p))\sqrt{\omega_m(p)}}$$

It follows that

$$\mathcal{U}(\Lambda\, 0)\,|(m\,j)\,p\,\mu\rangle_{cov} =$$

$$\sum_{\nu} |(m\,j)\,\Lambda p\,\nu\rangle_{cov}\, D^{j}_{\nu\mu}(\Lambda)$$

where here $\Lambda \in SL(2\mathbb{C})$. Note

that

$$I = \sum_{\mu}\int |(m\,j)\,\bar{p}\,\mu\rangle\, d^3p\, \langle (m\,j)\,\bar{p}\,\mu| =$$

$$\sum_{\mu\nu}\int |(m\,j)\,p\,\mu\rangle_{cov}\, \frac{D_{\mu\alpha}(B(p))\,D_{\alpha\nu}(B^{\dagger}(p))}{\omega_m(p)}\,{}_{\checkmark}$$

$$d^3p \quad {}_{cov}\langle (m\,j)\,p\,\mu|$$

since $B(p) = B^{\dagger}(p)$ and

$$B^2(p) = e^{\bar{p}\cdot\sigma} = \cosh p \pm \hat{p}\cdot\bar{\sigma}\sinh p$$

$$= \frac{p^0}{m}I + \bar{p}\cdot\frac{\bar{\sigma}}{m} = (p\cdot\sigma)/m$$

and $$\frac{d^3p}{\omega(p)} = \int d^4p\, \delta(p^2-m^2)\,\Theta(p)\,2$$

so this becomes

$$\boxed{I = \int d^4p \, |(m j) \bar{p} \mu \rangle_{cov} \, 2\delta(p^2+m^2)\theta(p^0) D^j_{\mu\nu}(\sigma \cdot p) \langle (m j) \bar{p} \nu |}$$

In this case the inner product for the covariant wave functions have a non trivial kernel

$$2\delta(p^2+m^2)\theta(p^0) D^j_{\mu\nu}(\sigma \cdot p)$$

For rotations since $R^\dagger = R^{-1} \Rightarrow R = (R^\dagger)^{-1}$, however for general $SL(2C)$ matrices

$$A^{-1} \neq A^\dagger$$

This means that there is another way to define covariant states

$$\boxed{\begin{aligned} &|(m\tfrac{1}{2})\,p\,\mu\rangle_{\overline{cov}} = \\ &\sum_{\nu} |(m\tfrac{1}{2})\,p\,\nu\rangle\, D^{\frac{1}{2}}_{\nu\mu}(B^{+}(p))\,\sqrt{\omega_m(p)} \end{aligned}}$$

which transforms like

$$\boxed{\begin{aligned} &\mathcal{U}(\Lambda_0)\,|(m\tfrac{1}{2})\,p\,\mu\rangle_{\overline{cov}} = \\ &\sum_{\nu} |(m\tfrac{1}{2})\,\Lambda p\,\nu\rangle_{\overline{cov}}\, D^{\frac{1}{2}}_{\nu\mu}((\Lambda^{+})^{-1}) \end{aligned}}$$

both Λ and $(\Lambda^{+})^{-1}$ are representations of $SL(2C)$, but there is <u>no</u> similarity transformation S satisfying

$$S\Lambda S^{+} = (\Lambda^{+})^{-1}$$

for all Λ.

not $(A^{+})^{-1} = \left(e^{-\bar{z}^{*}\cdot\bar{\sigma}}\right) =$

$$= e^{z^{*} \sigma_2 \sigma^{*} \sigma_2} = \sigma_2 A^{*} \sigma_2$$

$$X' = A X A^{+}$$

$$\sigma_2 X'^{*} \sigma_2 = (A^{+})^{-1} \sigma_2 X^{*} \sigma_2 A^{-1}$$

preserves $\det(\sigma_2 X^{*} \sigma_2) = x^{0\,2} - \bar{x}^2$

not $\sigma_2 X^{*} \sigma_2 = \begin{pmatrix} x^0 - x_3 & -x^1 + i x^2 \\ -x^1 - i x^2 & x^0 + x_3 \end{pmatrix}$

which corresponds to a space reflection. we define

$$\Pi \cdot (p^0, \bar{p}) = (p^0, -\bar{p})$$

$$\Pi (x^0, \bar{x}) = (x^0, -\bar{x})$$

$$I = \sum_{\mu\nu} \int | (m \tfrac{1}{2}) p \mu \rangle_{cov}^{x} d^4 p\, \delta(p^2 + m^2)\, \theta(p) \; D^{\frac{1}{2}}_{\mu\nu} (\sigma \cdot \Pi p) \langle_{cov} (m \tfrac{1}{2}) p \nu |$$

Because of the role of space reflection $\Lambda, (\Lambda^{\dagger})^{-1}$ are called "right" and left handed representation of $SL(2,\mathbb{C})$

∴ If we want to have space reflection as a linear transformation in a covariant representation we need a direct sum of right and left handed representations.

To do this we proceed as in the Poincare covariant case.. we start with a 4 component state at

We define Dirac spinors by

$$u(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The first column is spin up in the rest frame while the second column is spin down in the rest fram

$$u(p) = \begin{pmatrix} D^{1/2}(B(p/m)) & 0 \\ 0 & D^{1/2}((B^{+}(\frac{p}{m}))^{-1}) \end{pmatrix} u(0)$$

note

$$D^{1/2}_{\mu\nu}(B(p/m)) = B(p/m)_{\mu\nu} =$$

$$\cosh \tfrac{\rho}{2}\, I + \hat{p}\cdot\bar{\sigma} \sinh \tfrac{\rho}{2} =$$

$$\sqrt{\frac{\cosh\rho + 1}{2}}\, I + \hat{p}\cdot\bar{\sigma}\sqrt{\frac{\cosh\rho - 1}{2}}$$

$$= \sqrt{\frac{p^0 + mc}{2mc}} + \sqrt{\frac{p^0 - mc}{2mc}}\, \hat{p} \cdot \bar{\sigma}$$

$$= \frac{1}{\sqrt{2mc(p^0 + mc)}} \left((p^0 + mc) I + \bar{p} \cdot \bar{\sigma} \right)$$

while

$$B^{\dagger}(p)^{-1} = \frac{1}{\sqrt{2mc(p^0 + mc)}} \left((p^0 + mc) I - \bar{p} \cdot \bar{\sigma} \right)$$

This gives

$$\boxed{u(p) = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2mc(p^0 + mc)}} \begin{pmatrix} p^0 + mc + p^3 & p_1 - i p_2 \\ p^1 + i p^2 & p^0 + mc - p^3 \\ p^0 + mc - p^3 & -p_1 + i p_2 \\ -p_1 - i p^2 & p^0 + mc + p_3 \end{pmatrix}}$$

This is called a Dirac u spinor

To construct a Lorentz covariant equation for $u(p)$ define

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \pi\sigma_\mu & 0 \end{pmatrix}$$

$$\begin{pmatrix} \Lambda & 0 \\ 0 & (\Lambda^+)^{-1} \end{pmatrix} \begin{pmatrix} 0 & \sigma_\mu \\ \pi\sigma_\mu & 0 \end{pmatrix} \begin{pmatrix} \Lambda^{-1} & 0 \\ 0 & \Lambda^+ \end{pmatrix} =$$

$$\begin{pmatrix} \Lambda & 0 \\ 0 & (\Lambda^+)^{-1} \end{pmatrix} \begin{pmatrix} 0 & \sigma_\mu \Lambda^+ \\ \pi\sigma_\mu \Lambda^{-1} & 0 \end{pmatrix} \doteq$$

$$\begin{pmatrix} 0 & \Lambda\sigma_\mu\Lambda^+ \\ (\Lambda^+)^{-1}\pi\sigma_\mu\Lambda^{-1} & 0 \end{pmatrix} = \sum_\nu \gamma_\nu \Lambda^\nu{}_\mu$$

we define $S(\Lambda) = \begin{pmatrix} \Lambda & 0 \\ 0 & (\Lambda^+)^{-1} \end{pmatrix}$

$$S(\Lambda)\gamma_\mu S(\Lambda^{-1})^{-1} = \sum \gamma_\nu \Lambda^\nu{}_\mu$$

we can also define

$$\gamma^\mu = \begin{pmatrix} 0 & -\pi\sigma_\mu \\ -\sigma_\mu & 0 \end{pmatrix}$$

direct calculation shows

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2\eta^{\mu\nu}$$

using this we calculate

$$(\sum p^\mu \gamma_\mu - mc)\, u(p) =$$

$$(\sum p^\mu \gamma_\mu - mc)\, S(B(p))\, u(0) =$$

$$S(B(p))\, S(B(p))^{-1} (\sum p^\mu \gamma_\mu - mc)\, S(B(p))\, u(0)$$

$$S(B(p))\, (\sum p^\mu (\gamma_\nu (B^{-1}(p))^\nu{}_\mu - mc)\, u(0) =$$

$$S(B(p))\, (\sum B(p)^{-1\,\nu}{}_\mu\, p^\mu\, \gamma_\nu - mc)\, u(0)$$

$$S(B(p))\, (mc\gamma_0 - mc)\, u(0)$$

$$mc \begin{pmatrix} -I & I \\ I & -I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = 0$$

If we use $p^\mu = \sum \eta^{\mu\nu} \left(\frac{\hbar}{i} \frac{\partial}{\partial x^\nu} \right)$

and let $\Psi(x,t) = u(p)\, e^{ip\cdot x/\hbar}$

we see

$$\left(\frac{\hbar}{i} \frac{\partial}{\partial x_\mu} \eta^{\mu\nu} \gamma_\nu - mc \right) \Psi(xt) = 0$$

This is called the Dirac equation.

just like with the Klein Gordan equation these are not the only solutions

consider

$$V(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$V(p) = \begin{pmatrix} B(p) & 0 \\ 0 & B^{-1}(p) \end{pmatrix} V(0)$$

If we let $p^0 = mc \to -mc$ then

$$(p^\mu \gamma_\mu - mc)_{rest} =$$

$$mc \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I \\ -I \end{pmatrix} = 0$$

this gives negative energy solutions

Since this is an equation for spin $\frac{1}{2}$ Dirac proposed that all of the negative energy states were filled. Absence of one of the filled particles behaves like a positively charged electron → this led to the prediction of the positron

But this fails because a negative energy proton and negative energy electron can make negative energy Hydrogen atoms which are bosons.

D = correct answer – the Dirac equation is a classical field equation that must be quantized.