scattering

$$\rho_b = \sum_n |\psi_{bn}\rangle P_{bn} \langle \psi_{bn}| \qquad \langle \bar{P}_b \rangle = Tr(\bar{P}\rho_B)$$

$$\rho_T = \sum_m |\psi_{Tm}\rangle P_{Tm} \langle \psi_{Tm}| \qquad \langle \bar{P}_T \rangle = Tr(\bar{P}\rho_T)$$

single scattering

Probability

$$P_{fi} = |\langle \psi_+(t) | \psi_-(t) \rangle|^2$$

independent of time

$|\psi_+(t)\rangle$ - detector

$|\psi_-(t)\rangle$ - beam + target

Asymptotic condition

$$\lim_{t \to \pm\infty} \| e^{-iHt/\hbar} |\psi_\pm(0)\rangle - e^{-iH_0t/\hbar} |\psi^0_\pm(0)\rangle \| = 0$$

$$|\psi_\pm(0)\rangle = \lim_{t \to \pm\infty} e^{iHt/\hbar} e^{-iH_0t/\hbar} |\psi^0_\pm(0)\rangle$$

* sufficient condition for existence of limit

$$\int_a^\infty \| V e^{\pm iH_0t/\hbar} |\psi^0(0)\rangle \| \, dt < \infty$$

$$\Omega_\pm = \lim_{t \to \pm\infty} e^{iHt/\hbar} e^{-iH_0t/\hbar}$$

consequences

(1) $e^{-iHt}\,\Omega_\pm = \Omega_\pm\, e^{-iH_0 t}$

(2) $H\,\Omega_\pm = \Omega_\pm\, H_0$

(3) $|\psi_\pm(t)\rangle = e^{-iHt/\hbar}\,\Omega_\pm\,|\psi^0_\pm(0)\rangle =$

$= \Omega_\pm\, e^{-iH_0 t/\hbar}\,|\psi^0_\pm(0)\rangle$

$= \Omega_\pm\,|\psi^0_\pm(t)\rangle$

(4) $0 = \langle b|\,\Omega_\pm\,|\psi^0_\pm(0)\rangle = \langle b|\psi_\pm(0)\rangle = 0$

$\langle b|\,\Omega_\pm = 0$

(5) $P = |\langle\psi_+(0)|\psi_-(0)\rangle|^2 =$

$|\langle\psi^0_+(0)|\underbrace{\Omega_+^\dagger\,\Omega_-}_{S}|\psi^0_-(0)\rangle|^2$

$S = \Omega_+^\dagger\,\Omega_-$

(6) $S^\dagger S = I$ unitarity

(7) $H_0 S = S H_0$

calculation of S

$\langle\bar{p}_1'\bar{p}_2'|\,S\,|\bar{p}_1\bar{p}_2\rangle = \langle\bar{p}_1'\bar{p}_2'|\bar{p}_1\bar{p}_2\rangle$

$-\,2\pi i\,\delta(E_1'+E_2'-E_1-E_2)\,\langle\bar{p}_1'\bar{p}_2'|\,T(E+i\epsilon)\,|\bar{p}_1\bar{p}_2\rangle$

$E = E_1 = E_2$

$$T(z) = V + V(z-H)^{-1}V$$

$$T(z) = V + V(z-H_0)^{-1}T(z)$$

(Lippmann Schwinger Equation)

$$(z-H)^{-1} = (z-H_0)^{-1} + (z-H_0)^{-1}V(z-H)^{-1}$$
$$= (z-H_0)^{-1} + (z-H)^{-1}V(z-H_0)^{-1}$$

2nd resolvent identity

Born series

$$T(z) = V + \sum_{n=1}^{\infty} (V(z-H_0)^{-1})^n V$$

converges when

$$\| V(z-H_0)^{-1} \| < 1$$

fails when H has bound states

Born approximation

$$\langle p_1' p_2' | T(E+i\epsilon) | p_1 p_2 \rangle =$$

$$\delta(\bar{P}_1' + \bar{P}_2' - \bar{P}_1 - \bar{P}_2) \langle p_1' p_2' \| T(E+i\epsilon) \| p_1 p_2 \rangle$$

$$p_1 p_2 \rightarrow \bar{P}\, \bar{k}$$

$$\delta(\bar{P}' - \bar{P}) \langle \bar{k}' \| T(\mathcal{E}+i\epsilon) \| \bar{k} \rangle \qquad \mathcal{E} = \frac{k^2}{2\mu} = \frac{k'^2}{2\mu}$$

Born approx

$$\langle \bar{k}' \| T(\mathcal{E}+i\epsilon) \| \bar{k} \rangle \simeq \langle \bar{k}' | V | \bar{k} \rangle =$$

$$\frac{1}{(2\pi\hbar)^3}\int e^{-i(\bar{k}'-\bar{k})\cdot\bar{r}}\, V(r)\, d^3r$$

$$\frac{1}{(2\pi\hbar)^{3/2}}\tilde{V}(\bar{k}'-\bar{k}) \quad \text{(Fourier transform)}$$

Other methods

$V(z-H_0)^{-1}$ compact $\Rightarrow$

$$V(z-H_0)^{-1} = \sum_{mn}^{N} |m\rangle v_{mn}\langle n| + \Delta = F+\Delta$$

Δ can be made small as desired choosing large enough F

$$T = V + (F+\Delta)T =$$

$$(1-F)T = V + \Delta T$$

$$T = (1-F)^{-1}V + (1-F)^{-1}\Delta T$$

$$= (1-F)^{-1}V + \underbrace{\sum_{n=1}^{\infty}\left((1-F)^{-1}\Delta\right)^n (1-F)^{-1}V}_{\text{converges}}$$

$$\langle n|T|k\rangle = \langle n|V|k\rangle + \sum\langle n|V(z-H_0)^{-1}|m\rangle\langle m|T|k\rangle$$

$$= \left(\delta_{nm} - \langle n|V(z-H_0)^{-1}|m\rangle\right)^{-1}\cdot\langle m|V|k\rangle$$

$$\langle\bar{k}'|T|\bar{k}\rangle = \langle\bar{k}'|V|\bar{k}\rangle + \sum_m\langle\bar{k}'|V(z+H_0)^{-1}|m\rangle\langle m|T|k\rangle$$

Yukawa - Debye Huckle

$\frac{\lambda e^{-\alpha r}}{r}$ Born Approximation $\rightarrow$ Coulomb result as $\alpha \rightarrow 0$

$$\sqrt{\frac{1-\cos\theta}{2}} = \sin\left(\frac{\theta}{2}\right)$$

differential cross section - used density matrices

$$\frac{dN}{dt\,dV} \propto n_T\, n_b \bar{v} \times d\sigma$$

$$d\sigma = \frac{(2\pi)^4}{V} \hbar^2 \left|\langle \bar{p}_1' \bar{p}_2' \| T(E+i\epsilon) \| \bar{p}_1 \bar{p}_2 \rangle\right|^2 \times$$

$$\delta(E_1' + E_2' - E_1 - E_2)\, \delta(\bar{p}_1' + \bar{p}_2' - \bar{p}_1 - \bar{p}_2)\, d^3p_1\, d^3p_2$$

Integrate over all unmeasured variables - use δ functions

CM integrate $d^3p \frac{dk}{dE} dE$

Lab integrate $d^3p_2' \frac{d|p_1|}{dE} dE$

$$E = \frac{p_1^2}{2m_1} + \frac{(p-p_1)^2}{2m_2} = \frac{p_1^2}{2\mu} + \frac{p^2}{2m_2} - \frac{p_1 p \cos\theta}{m_2}$$

Scattering amplitude

$$F = -(2\pi)^2 \hbar \mu \langle \bar{k}' \| T(E+i\epsilon) \| \bar{k} \rangle$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = |F(k'k)|^2$$

Dyson series

$$\mathcal{U}_I(t\,t') = e^{iH_0 t/\hbar}\, e^{-iH(t-t')/\hbar}\, e^{-iH_0 t'/\hbar}$$

$$\frac{d\mathcal{U}_I}{dt} = -\frac{i}{\hbar} V_I(t)\,\mathcal{U}_I(t\,t') \qquad \mathcal{U}_I(t'\,t') = I$$

$$\mathcal{U}_I(t\,t') = I + \sum_{n=1}^{\infty} \left(-\frac{i}{\hbar}\right)^n \frac{1}{n!} \int_{t'}^{t} T(V_I(t_1) \cdots V_I(t_n))\, dt_1 \, dt_n$$

$$S = \lim_{\substack{t \to +\infty \\ t' \to -\infty}} \mathcal{U}_I(t\,t') =$$

$$= I + \sum_{n=1}^{\infty} \left(-\frac{i}{\hbar}\right)^n \frac{1}{n!} \int_{-\infty}^{\infty} T(V_I(t_1) \cdots V_I(t_n))\, dt_1 \, dt_n$$

Born Approximation

$$S - I \simeq -\frac{i}{\hbar} \int_{-\infty}^{\infty} V_I(t)\, dt = -i/\hbar \int_{-\infty}^{\infty} e^{iH_0 t/\hbar}\, V\, e^{-iH_0 t/\hbar}$$

matrix elements

$$\langle k' | \left(-\frac{i}{\hbar} \int_{-\infty}^{\infty} e^{iH_0 t/\hbar}\, V\, e^{-iH_0 t/\hbar}\right) | k \rangle =$$

$$\left(-\frac{i}{\hbar}\right)(2\pi)\hbar \;\delta(E-E') \langle \bar{k}' | V | \bar{k} \rangle =$$

$$-2\pi i \;\delta(E-E') \langle \bar{k}' | V | k \rangle$$

Wave functions

$$\langle \bar{r} | k_- \rangle = \langle \bar{r} | \bar{k} \rangle + \int \underbrace{\langle \bar{r} | (E - H_0 + i\epsilon) | r' \rangle}_{-\frac{\mu}{2\pi\hbar^2} \frac{e^{\pm ik|r-r'|/\hbar}}{|r-r'|}} V(r')\, d^3r' \langle \bar{r}' | \bar{k} \rangle$$

Large r

$$\to \langle \bar{r} | \bar{k} \rangle - \frac{\mu}{2\pi\hbar^2} \frac{e^{\pm ikr/\hbar}}{r} \int e^{\mp ik\hat{r}\cdot\bar{r}'/\hbar}\, V(\bar{r}') \langle \bar{r}' | \bar{k}_- \rangle\, d^3r'$$

note

$$\langle \bar{k}' | V | \bar{k}_- \rangle =$$

$$\langle \bar{k}' | T(E+i\epsilon) | \bar{k} \rangle =$$

$$\langle \bar{k}'_+ | V | \bar{k} \rangle$$

$$\langle \bar{r} | k_- \rangle \to \frac{1}{(2\pi\hbar)^{3/2}} \left[e^{i\bar{k}\cdot\bar{r}/\hbar} - \overbrace{\frac{\mu}{2\pi\hbar^2}(2\pi\hbar)^3}^{-(2\pi)^2\mu\hbar} \frac{e^{\pm i\frac{kr}{\hbar}}}{r} \times \right.$$

$$\underbrace{\frac{1}{(2\pi\hbar)^{3/2}} \int e^{\mp i \frac{\hat{r}k\cdot\bar{r}'}{\hbar}} V(r') \, d^3r' \langle \bar{r}' | k_- \rangle}_{\underbrace{\langle k\hat{r} | V | k_- \rangle}_{\langle k\hat{r} | T | \bar{k} \rangle}}$$

$$\to \frac{1}{(2\pi\hbar)^{3/2}} \left[e^{i k\cdot\bar{r}/\hbar} \underbrace{- (2\pi)^2\mu\hbar \langle \hat{r}k | T | k \rangle}_{F} \frac{e^{ikr}}{r} \right]$$

$$F(\hat{k}'k) = -(2\pi)^2\mu\hbar \langle \bar{k}' || T(E+i\epsilon) || k \rangle$$

optical theorem

$$S S^\dagger = I \qquad |E\theta\phi\rangle = \sqrt{\mu k}\, |k\theta\phi\rangle$$

$$\mathrm{Im}\,(F(\hat{k}\hat{k})) = \frac{\hbar k}{4\pi} \sigma_T$$

$$\sigma_T = \frac{4\pi}{\hbar k} \mathrm{Im}\,(F(\hat{k};\hat{k}))$$

Rotationally invariant potentials
partial waves

$$\langle \bar{k}' | T(E+i\epsilon) | \bar{k} \rangle = \sum_{\ell m} Y_{\ell m}(\hat{k}') \, t_\ell(k', E+i\epsilon, k) \, Y^*_{\ell m}(\hat{k})$$

$$F(\bar{k}', \bar{k}) = \sum_{\ell m} Y_{\ell m}(\hat{k}') \, f_\ell(k) \, Y^*_{\ell m}(\hat{k})$$

$$\langle \bar{k}' | V | \bar{k} \rangle = \sum_{\ell m} Y_{\ell m}(\hat{k}') \, V_\ell(k' k) \, Y^*_{\ell m}(\hat{k})$$

$$\langle \bar{r} | \bar{k} \rangle = \frac{1}{(2\pi\hbar)^{3/2}} \sum 4\pi \, i^\ell \, j_\ell\left(\frac{kr}{\hbar}\right) Y_{\ell m}(\hat{r}) Y^*_{\ell m}(\hat{k})$$

$$P_\ell(\hat{k}' \cdot \hat{k}) = \sum \frac{4\pi}{2\ell+1} Y_{\ell m}(\hat{k}') Y^*_{\ell m}(\hat{k})$$

$$j_\ell(x) = \frac{(-i)^\ell}{2} \int_{-1}^{1} e^{ipx} P_\ell(p) \, dp$$

$$\langle r' | (E - h_0 + i\epsilon)^{-1} | r \rangle = -\frac{2\mu k}{\hbar^3} \sum Y_{\ell m}(\hat{r}') \, j_\ell\left(\frac{kr_<}{\hbar}\right) h^+_\ell\left(\frac{kr_>}{\hbar}\right) Y^*_{\ell m}(\hat{r})$$

Lippmann Schwinger equation for wave function

$$t_\ell(k' k \, E+i\epsilon) = V_\ell(k' k) + \int_0^\infty V_\ell(k' k'') \frac{k''^2 dk''}{k^2/2\mu - \frac{k''^2}{2\mu} + i\epsilon} t_\ell(k'', k, E+i\epsilon)$$

$$\langle r_\ell | k^-_\ell \rangle = \langle r_\ell | k_\ell \rangle - \frac{2\mu k}{\hbar^3} \int_0^\infty j_\ell\left(\frac{kr_<}{\hbar}\right) h_\ell\left(\frac{kr_>}{\hbar}\right) V(r') \, r'^2 \, dr' \, \langle r' | k_\ell \rangle$$

energy eigenstates $\quad S = e^{2i\delta}$

$$|E\ell m\rangle = |k\ell m\rangle \sqrt{k\mu}$$

$$S_\ell(E) = e^{2i\delta_\ell(E)} = 1 - 2\pi i k\mu\, t_\ell$$

$$\left.\begin{aligned} e^{i\delta_\ell}\sin\delta_\ell &= -\pi\mu k\, t_\ell \\ f_\ell &= -(2\pi)^2 \mu\hbar\, t_\ell \end{aligned}\right\}$$

$$f_\ell = -(2\pi)^2\mu\hbar\left(\frac{-1}{\pi\mu k}\right)e^{i\delta_\ell}\sin\delta_\ell$$

$$= \frac{4\pi\hbar}{k}\, e^{i\delta_\ell}\sin\delta_\ell$$

$$\boxed{\sigma_T = \sum \frac{4\pi\hbar^2}{k^2}(2\ell+1)\sin^2\delta_\ell}$$

phase shift $\quad \langle r, | k, \rangle \rightarrow \frac{4\pi i^\ell}{(2\pi\hbar)^3} \frac{\sin(kr/\hbar - \frac{\ell\pi}{2} + \delta_\ell)}{(kr/\hbar)} \cdot e^{i\delta_\ell}$

Spin $\quad \frac{d\sigma}{d\Omega} = \mathcal{M}\, \rho_B \otimes \rho_T\, \mathcal{M}^\dagger$

$$\mathcal{O} = \frac{\mathrm{Tr}(\mathcal{O}\mathcal{M}\rho_B\otimes\rho_T\mathcal{M}^T)}{\mathrm{Tr}(\mathcal{M}\rho_B\otimes\rho_T\mathcal{M}^T)}$$

Identical particles

$$F(\vec{k}', k) \rightarrow F(\vec{k}', k) \pm F(-\vec{k}', k)$$

$$\rho = \frac{I + \vec{P}\cdot\vec{\sigma}}{2} \qquad P = \text{polarization}$$

Inverse scattering

$$A H A^{\dagger} = H'$$

$$\lim_{t \to \pm\infty} \| (1 - A) e^{-iH_0 t/\hbar} |\psi\rangle \| = 0$$

$$\Updownarrow$$

$$S = S'$$

For $A = I + \Delta$

$$(I + \Delta)(H_0 + V)(I + \Delta^{\dagger}) =$$

$$H_0 + \underbrace{V + \Delta H + H\Delta^{\dagger} + \Delta H \Delta^{\dagger}}_{V'}$$