Lecture 34

Poincaré group

$$x^\mu \to x^{\mu'} = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu x^\nu + a^\mu$$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$x^\mu = (ct, x, y, z)$$

$$\eta_{\mu\nu} = \sum_{\alpha\beta} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \eta_{\alpha\beta}$$

$\Lambda^\mu{}_\nu$, a^μ constant

These transformations relate different inertial coordinate reference frames

Relativistic Quantum Mechanics

$$|\psi\rangle \to |\psi'\rangle = U(\Lambda a)|\psi\rangle$$

$$\mathcal{U}(\Lambda_2 a_2)\,\mathcal{U}(\Lambda_1 a_1) = \mathcal{U}(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2)$$

$$\mathcal{U}^{+}(\Lambda a) = \mathcal{U}^{-1}(\Lambda a)$$

preserves all quantum observables in different inertial coordinate systems

$SL(2\,\mathbb{C})$

$$X = \sum x^{\mu}\sigma_{\mu} \qquad X = X^{+}$$

$$\sigma_{\mu} = (I, \bar{\sigma}) \qquad \det X = (x^0)^2 - |\vec{x}|^2$$

$$x^{\mu} = \frac{1}{2}\,\mathrm{Tr}(\sigma_{\mu} X)$$

$$X \to X' = AXA^{+} + B$$

$$\det A = 1 \qquad B = B^{+} = \text{constant}$$

most general A

$$A = \pm\, e^{\bar{z}\cdot\bar{\sigma}}$$

$$\bar{z} = \left(\frac{\bar{p}}{2} - i\frac{\bar{\theta}}{2}\right) \quad \text{complex vector}$$

$$\underset{\sim}{P} = e^{\frac{\bar{p}}{2}\cdot\bar{\sigma}} = \cosh\left(\frac{p}{2}\right) + \hat{p}\cdot\bar{\sigma}\sinh\left(\frac{p}{2}\right)$$

$$\underset{\sim}{u} = e^{-i\frac{\bar{\theta}}{2}\cdot\bar{\sigma}} = \cos\left(\frac{\theta}{2}\right) - i\,\bar{\sigma}\cdot\hat{\theta}\sin\left(\frac{\theta}{2}\right)$$

Polar decomposition

$$A = PU = U'P'$$

construction of $U(4a)$

for an electron

* state of electron fixed
  by measuring
  linear momentum
  and spin projection

$$|(m\tfrac{1}{2})\,\bar{p}\,u\rangle \quad (\text{basis})$$

$$\psi(\bar{p}, u) = \langle (m\tfrac{1}{2})\,\bar{p}\,u \,|\, \psi\rangle$$

$\mathcal{U}(I\, a)\, \mathcal{U}(\Lambda\, 0) =$

$\mathcal{U}(I\Lambda, I \cdot 0 + a) = \mathcal{U}(\Lambda\, a)$

$*$ $\mathcal{U}(R, 0)\, |(m \tfrac{1}{2})\, \bar{0}\, \mu\rangle$

a rotation leaves

$p^{\mu} = (mc, 0\,0\,0)$ unchanged

$\therefore$ $\mathcal{U}(R, 0)\, |(m \tfrac{1}{2})\, |\bar{0}\, \mu\rangle =$

$$\sum_{\nu = -1/2}^{1/2} |(m \tfrac{1}{2})\, \bar{0}\, \nu\rangle\, \mathcal{M}_{\nu\mu}(R)$$

$\mathcal{M}_{\mu\nu}(R)$ must be a

2×2 unitary matrix

satisfying $\mathcal{M}(R_2)\, \mathcal{M}(R_1) =$

$\mathcal{M}(R_2 R_1)$

In the basis of eigenstate

so $\bar{J} \cdot \hat{z}$ $\mathcal{M}(R) = D^{\frac{1}{2}}_{\mu\nu}(R)$

It follows that for

an electron at rest

$$\boxed{\mathcal{U}(R\,0)\,|(m\tfrac{1}{2})\,\bar{0}\,\mu\rangle = \sum_{\nu=-1/2}^{1/2} |(m\tfrac{1}{2})\,\bar{0}\,\nu\rangle\, D^{1/2}_{\nu\mu}(R)}$$

where $\underline{\underline{R}}$ is an SU(2) matrix $R = e^{-i\frac{\bar{\theta}}{2}\cdot\bar{\sigma}}$

* Next consider the case where $A = e^{\frac{\bar{\rho}}{2}\cdot\bar{\sigma}}$

note

$$\cosh\rho = \frac{p^0}{mc}$$

$$\sinh\rho = \frac{|\vec{p}|}{mc}$$

define

$$B(p/m) = e^{\frac{\bar{\rho}}{2}\cdot\bar{\sigma}} =$$

Rotationless Lorentz boost

$$\Lambda(B(p/m))\,(mc,0,0,0) = p^{\mu}$$

$\mathcal{U}(B(p), 0)\, |(m\tfrac{1}{2})\, \bar{0}, \mu\rangle$

* gives a state with momentum $\bar{p}$
* must be unitary

remark –

$\Lambda = B(p)\, R(p)$

$\mathcal{U}(\Lambda, 0)\, |(m\tfrac{1}{2})\, 0\, \mu\rangle$

will also

* give a state with momentum $\bar{p}$
* must be unitary

we define

$|(m\tfrac{1}{2})\, \bar{p}\mu\rangle \equiv$

$\mathcal{U}(B(p), 0)\, |(m\tfrac{1}{2})\, \bar{0}, \mu\rangle\, N(p)$

* The states with non 0 momentum are defined so $\mathcal{U}(B(\vec{p}))$ does not change σ

* The factor $N(p)$ is chosen so this transformation is unitary

* The interpretation of σ is the value of J_z that would be measured in the electron's rest frame if it was transformed to the rest frame with $B^{-1}(\vec{p})$

construction of N

Note $\int d^4p\, \delta(p^2+m^2)\,\Theta(p^0)$

is invariant under

Lorentz transformation

$$d^4p = d^4p' \qquad \det\Lambda = 1$$

$$p^{0'} > 0 \qquad p^0 > 0 \qquad \Lambda^0{}_0 \geq 1$$

$$p^2 = \sum \eta_{\mu\nu} p^\mu p^\nu = \sum \eta_{\mu\nu} p'^\mu p'^\nu$$

$$\int d^4p\, \delta(p^2+m^2c^2)\,\Theta(p^0) =$$

$$\int d^3p \frac{1}{\sqrt{\bar{p}^2+m^2c^2}} = \int d^3p' \frac{1}{\sqrt{\bar{p}'^2+m^2c^2}}$$

define $\omega_m(\bar{p}) = \sqrt{\bar{p}^2+m^2c^2}$

$$I = \sum |(m\tfrac{1}{2})\, p\, u\rangle\, d^3p\, \langle (m\tfrac{1}{2})\, p\, u|$$

$$= \sum |(m\tfrac{1}{2})\, p'\, u\rangle\, d^3p'\, \langle (m\tfrac{1}{2})\, \bar{p}'\, u|$$

$$\frac{d^3 p}{2\omega_m(p)} = \frac{d^3 p'}{2\omega_m(p')}$$

$$d^3 p = d^3 p' \frac{\omega_m(p)}{\omega_m(p')}$$

If we choose to normalize states so

$$\langle (m \tfrac{1}{2})\, \bar{p}' \mu' \,|\, (m \tfrac{1}{2})\, \bar{p} \mu \rangle$$

$$= \delta(\bar{p}' - \bar{p})\, \delta_{\mu' \mu} =$$

$$\langle (m \tfrac{1}{2})\, \bar{p}' \mu' \,|\, U^{\dagger}(\Lambda, 0)\, U(\Lambda, 0) \,|\, (m \tfrac{1}{2})\, \bar{p}$$

$$N(p)^2\, \delta(\Lambda \bar{p} - \Lambda p')\, \delta_{\mu' \mu}$$

$$N(p)^2 \left| \frac{d^3 p}{d^3 p'} \right| \delta(p - p')\, \delta_{\mu' \mu}$$

$$N(p)^2 \frac{\omega(p)}{\omega(\Lambda p)}\, \delta(p - p')\, \delta_{\mu \mu}$$

$$\therefore \quad \boxed{N(p) = \sqrt{\frac{\omega(\Lambda p)}{\omega(p)}}}$$

$$|(m\tfrac{1}{2})\,\bar{p}\,\mu\rangle = \mathcal{U}(B(p),0)\,|(m\tfrac{1}{2})\,\bar{0}\,\mu\rangle \sqrt{\frac{m}{\omega_m(\bar{p})}}$$

* translations

$$\mathcal{U}(I,a)\,|(m\tfrac{1}{2})\,\bar{0}\,\mu\rangle = e^{-ia^0mc/\hbar}\,|(m\tfrac{1}{2})\,\bar{0}\,\mu\rangle$$

A general $\mathcal{U}(\Lambda,a)$ can be expressed in terms of the 3 unitary transformations acting on $|(m\tfrac{1}{2})\,\bar{0}.\mu\rangle$

$$\mathcal{U}(\Lambda\, a)\,|(m\tfrac{1}{2})\,\bar{p}\mu\rangle$$

$$\mathcal{U}(I,a)\,\mathcal{U}(\Lambda,0)\,\mathcal{U}(B(p),0)\,|(m\tfrac{1}{2})\,\bar{0}\,\mu\rangle\sqrt{\frac{m}{\omega_m(p)}} =$$

$$\mathcal{U}(I,a)\,\mathcal{U}(B(\Lambda p),0)\,\mathcal{U}(B^{-1}(\Lambda p),0)\,\times$$

$$\mathcal{U}(\Lambda,0)\,\mathcal{U}(B(p))\,|(m\tfrac{1}{2})\,\bar{0}\,\mu\rangle\sqrt{\frac{m}{\omega_m p)}} =$$

Note

$$B^{-1}(\Lambda p)\,\Lambda\, B(p) := R_w(\Lambda p)$$

$$0 \to p \to \Lambda p \to 0$$

this is a rotation called a canonical spin Wigner rotation

$$\mathcal{U}(\Lambda a)\,|(m\tfrac{1}{2})\,p\,\mu\rangle =$$

$$\mathcal{U}(I,a)\,\mathcal{U}(B(\Lambda p),0)\,\mathcal{U}(R_w(\Lambda p))\,|(m\tfrac{1}{2})\,\bar{0}\,\mu\rangle \sqrt{\frac{m}{\omega_m(p)}} =$$

$$\mathcal{U}(B(\Lambda p,0)\,\mathcal{U}(I,B^{-1}(\Lambda p)a)\,\mathcal{U}(R_w(\Lambda p))\,\times$$

$$|(m\tfrac{1}{2})\,0\,\mu\rangle \sqrt{\frac{m}{\omega_m(\bar{p})}} =$$

$$\mathcal{U}(B(\Lambda p,0)\,e^{-imc\,B^{-1}(\Lambda p)^{0}_{\nu}a^{\nu}}\,|(m\tfrac{1}{2})\,\bar{0}\,\nu\rangle\, D^{\frac{1}{2}}_{\nu\mu}(R_w(\Lambda p))$$

$$(mc,\bar{0})\cdot(B^{-1}(\Lambda p)\,a) =$$

$$B(\Lambda p)(mc,\cdots)\cdot a = \Lambda p\cdot a$$

$$\boxed{\sum_{\nu} e^{i\Lambda p\cdot a} \, |(\tfrac{1}{2}m)\Lambda p, \nu\rangle \, D^{\frac{1}{2}}_{\nu\mu}(R_\omega(\Lambda p)) \sqrt{\frac{\omega(\Lambda p)}{\omega(p)}}}$$

This gives a unitary representation of the poincare group on the electron Hilbert space.

Note that $U(\Lambda a)$ includes time translation $\Lambda = I$, $a = (ct, \bar{0})$

$$\frac{d}{dt} \langle (m\tfrac{1}{2})\bar{p}\mu | \psi \rangle =$$

$$-i \frac{p_0 c}{\hbar} \langle (m\tfrac{1}{2}) | p\mu | \psi \rangle$$

where $p_0 c = E$ = energy of electron with momentum $\bar{p}$.

This called the relativistic Schrodinger equation.

* This is a valid equation put $p \to p' = p - eA$ (A = vector potential) gave the wrong electron magnetic moment

* Differentiating twice gives

$$\frac{1}{c^2}\frac{d^2}{dt^2}\langle (m\tfrac{1}{2})\bar{p}\mu | \psi\rangle = -\frac{P_0^2}{\hbar^2} | (m\tfrac{1}{2})\, p\mu | \psi\rangle$$

note $\frac{P_0^2}{\hbar^2} = \frac{E^2}{c^2\hbar^2} = \frac{p^2c^2 + m^2c^4}{c^2\hbar^2} = \frac{\bar{p}^2 + m^2c^2}{\hbar^2}$

using $p = -\frac{i}{\hbar}\nabla$

$$= -\nabla^2 + m^2c^2$$

$$\frac{\partial^2}{\partial x_0^2}\langle (m\tfrac{1}{2})\bar{p}\mu | \psi\rangle = (\nabla^2 - m^2c^2)\langle (m\tfrac{1}{2})\bar{p}\mu | \psi\rangle$$

This is called the Klein Gordon equation by introducing the second derivative introduces negative energies. In addition it is necessary to have a first order equation in order to preserve probabilities

$$\frac{\partial}{\partial t} \langle \psi | \phi \rangle = \langle \psi | i\hbar (H - H^{\dagger}) | \phi \rangle = 0$$

This also gave the wrong magnetic moment.

Dirac - wanted a first order equation in time, and an equation with the same order of spatial derivatives that transformed in a simple way under Lorentz Transformations consider

$$\mathcal{U}(\Lambda,0)\,|(m\tfrac{1}{2})\,\vec{p}\mu\rangle =$$

$$|(m\tfrac{1}{2})\,\Lambda\vec{p}\,\nu\rangle\, D^{\frac{1}{2}}_{\nu\mu}\left(B^{-1}(\Lambda p)\Lambda B(p)\right)\sqrt{\frac{\omega(\Lambda p)}{\omega(p)}}$$

* since (1) $D^{j}_{\mu\nu}(R)$ is a homogeneous polynomial of degree $2j$ in the components of R

* $R = e^{-\frac{i}{2}\vec{\theta}\cdot\vec{\sigma}}$ is an entire function of angles

$\therefore$ $D^{j}_{\mu\nu}(R)$ is an entire function of angles

$$\Rightarrow\ D^{j}_{\mu\nu}(R_2R_1) - \sum D^{j}_{\mu\alpha}(R_2)\,D^{j}_{\alpha\nu}(R_1) = 0$$

$$\sum \langle j_1\mu_1, j_2\mu_2 | j\mu\rangle D^{j}_{\mu\nu}(R)\langle j\nu | j_1\nu_1 j_2\nu_2\rangle -$$

$$D^{j_1}_{\mu_1\nu_1}(R)\,D^{j_2}_{\mu_2\nu_2}(R) = 0$$

These equations hold in the case of complex angles

$\therefore$ They hold for $SL(2C)$

$$U(\Lambda,0)\,|(m\tfrac{1}{2})\,p\,\mu\rangle =$$

$$\sum |(m\tfrac{1}{2})\,\Lambda p\,\nu\rangle\, D^{\frac{1}{2}}_{\nu\alpha}(B^{-1}(\Lambda p)) \times$$

$$D^{\frac{1}{2}}_{\alpha\beta}(\Lambda)\, D^{\frac{1}{2}}_{\beta\nu}(B(p))\,\sqrt{\frac{\omega(\Lambda p)}{\omega(p)}} \Longrightarrow$$

multiply both sides by

$$D^{\frac{1}{2}}_{\beta\nu}(B(p)^{-1})\,\sqrt{\omega_m(p)} \Longrightarrow$$

$$\sum U(\Lambda,0)\,|(m\tfrac{1}{2})\,p\,\mu\rangle\, D^{\frac{1}{2}}_{\mu\alpha}(B^{-1}(p))\sqrt{\omega(p)} =$$

$$\sum |(m\tfrac{1}{2})\,\Lambda p\,\beta\rangle\, D^{\frac{1}{2}}_{\beta\sigma}(B^{-1}(\Lambda p))\sqrt{\omega(\Lambda p)}\; D^{\frac{1}{2}}_{\sigma\alpha}(\Lambda)$$

we define

$$|(m\tfrac{1}{2})\bar{p}\mu\rangle_{cov} \equiv$$

$$\sum |(m\tfrac{1}{2})\bar{p}\nu\rangle D^{\frac{1}{2}}_{\nu\mu}(B^{-1}(p))\sqrt{\omega_m(p)}$$

with this notation

$$\boxed{\mathcal{U}(\Lambda,0)\,|(m\tfrac{1}{2})\bar{p}\mu\rangle_{cov} = \sum |(m\tfrac{1}{2})\overline{\Lambda p}\,\nu\rangle_{cov}\, D^{\frac{1}{2}}_{\nu\mu}(\Lambda)}$$

Note for SU(2) rotation:

$R = (R^{\dagger})^{-1}$ this is <u>not</u>

true for $SL(2\mathbb{C})$

Alternatively replacing

R by $(R^{\dagger})^{-1}$ gives

$$|(m\tfrac{1}{2})\,\bar{p}\,\mu\rangle_{cov*} =$$

$$\sum_{\nu} |(m\tfrac{1}{2})\,\bar{p}\,\nu\rangle\, P^{\frac{1}{2}}_{\nu\mu}\left(B^{\dagger}(p)\right)\sqrt{\omega(p)}$$

$$\boxed{U(\Lambda\, 0)\,|(m\tfrac{1}{2})\,\bar{p}\,\mu\rangle_{cov*} = |(m\tfrac{1}{2})\,\overline{\Lambda p}\,\nu\rangle_{cov} \cdot D^{\frac{1}{2}}_{\nu\mu}\left((\Lambda^{\dagger})^{-1}\right)}$$

Λ $\quad (\Lambda^{\dagger})^{-1}$

are inequivalent representation of SL(2C). This means there are no matrices satisfying

$$M \Lambda M^{-1} = (\Lambda^{\dagger})^{-1}$$

(M constant)

These are both independent spin $\frac{1}{2}$ representation of SL(2C)

$$P = p^\mu \sigma_\mu = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix}$$

$$\sigma_2 P^* \sigma_2 = p^\mu \sigma_2 \sigma_\mu^* \sigma_2$$

$$= p^\mu (\sigma_0, -\vec{\sigma})$$

$$= \begin{pmatrix} p^0 - p^3 & -p^1 + ip^2 \\ -p^1 - ip^2 & p^0 + p^3 \end{pmatrix}$$

$$\sigma_2 \Lambda^* \sigma_2 = \sigma_2 \left(e^{z \cdot \sigma}\right)^* \sigma_2 =$$

$$= e^{z^* \cdot \sigma_2 \sigma^* \sigma_2} = e^{-z^* \cdot \vec{\sigma}}$$

$$= ((\Lambda)^\dagger)^{-1}$$

In the rest frame both

$$P \text{ and } \sigma_2 P^* \sigma_2 = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

with the upper entry correspond. to spin up and the lower corresponding to spin down

We also see $\sigma_2 P^* \sigma_2$ is the space reflected momentum

$$\mathcal{U}(\Lambda)\begin{pmatrix} |(m\tfrac{1}{2})\ \bar{p}\ u\rangle \\ |(m\tfrac{1}{2})\ \bar{p}\ u'\rangle \end{pmatrix} =$$

$$\sum \begin{pmatrix} |(m\tfrac{1}{2})\ \overline{\Lambda p}\ \nu\rangle \\ |(m\tfrac{1}{2})\ \overline{\Lambda p}\ \nu'\rangle \end{pmatrix} \times \begin{pmatrix} \Lambda_{\nu u} & 0 \\ 0 & (\Lambda^{+})^{-1}_{\nu u} \end{pmatrix}$$

$$\sum \begin{pmatrix} |(m\tfrac{1}{2})\ \overline{\Lambda p}\ \nu\rangle \\ |(m\tfrac{1}{2})\ \Lambda p\ \nu'\rangle \end{pmatrix} \times S(\Lambda)$$

To construct the covariant state start from the particles rest fram

Recall

$$B(p) = \cosh\frac{\rho}{2} + \hat{p}\cdot\bar{\sigma}\sinh\frac{\rho}{2}$$

$$= \sqrt{\frac{1+\cosh\rho}{2}} + \hat{p}\cdot\bar{\sigma}\sqrt{\frac{\cosh\rho - 1}{2}}$$

$$= \sqrt{\frac{mc+p^{0}}{2mc}} + \hat{p}\cdot\sigma\sqrt{\frac{p^{0}-mc}{2mc}}$$

$$\sigma_2 B^{*}(p)\sigma_2 =$$

$$= \sqrt{\frac{mc+p^{0}}{2mc}} - \hat{p}\cdot\sigma\sqrt{\frac{p^{0}-mc}{2mc}}$$

$$B(p/m) = \frac{p^0 + mc + \hat{p}\,(p^{0\,2} - m^2c^2)^{1/2}\cdot\sigma}{\sqrt{2mc\,(p^0 + mc)}}$$

$$= \frac{p^0 + mc + \bar{p}\cdot\bar{\sigma}}{\sqrt{2mc\,(p^0 + mc)}}$$

$$\sigma_2 B^* \sigma_2 = \frac{p^0 + mc - \bar{p}\cdot\bar{\sigma}}{\sqrt{2mc\,(p^0 + mc)}}$$