Lecture 32

Special relativity

* Inertial coordinate systems – free particles move with constant velocity

* Experiment ⟹ different inertial coordinate systems are related by Poincare transformation

$$(x^0, \vec{x}) = (ct, \vec{x})$$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Poincare group =

group of transformations

that preserve

$$c^2 \tau_{AB}^2 = \sum_{\mu\nu} \eta_{\mu\nu} (X_A - X_B)^\mu (X_A - X_B)^\nu$$

Homework – these have the form

$$\boxed{\begin{aligned} x^\mu \rightarrow x'^\mu &= \sum_\nu \Lambda^\mu{}_\nu x^\nu + a^\mu \\ \eta_{\mu\nu} &= \sum_{\alpha\beta} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \eta_{\alpha\beta} \end{aligned}}$$

where $\Lambda^\mu{}_\nu$, a^μ are constants

$\Lambda^\mu{}_\nu$ is called a <u>Lorentz</u> transformation

Lorentz transformations can be classified by

$$\det \Lambda = \pm 1$$

$$\Lambda^0{}_0 \geq 1, \leq -1$$

The transformations relevant to special relativity are the transformations satisfying

$$\det \Lambda = 1, \quad \Lambda^0{}_0 \geq 1$$

this set contains the identity. The other sets contain space and time reflections which do not leave the weak interaction invariant.

4 vectors are quantities like x^μ that transform under Lorentz transformations like

$$\boxed{x^\mu \to x'^\mu = \sum_\nu \Lambda^\mu{}_\nu x^\nu}$$

since $\tau^2 c^2 = \sum_{\mu\nu} \eta_{\mu\nu} x^\mu x^\nu$

is an invariant quantity

$$\frac{dx^\mu}{d\tau} \; , \; \frac{d^2x^\mu}{d\tau^2} \quad \frac{d}{d\tau}(m x^\mu) = p^\mu$$

(4 velocity, 4 acceleration, 4 momenta)

are all 4 vectors

note

$$\frac{dx^\mu}{d\tau} = \left(\frac{dx^0}{d\tau}, \frac{d\bar{x}}{d\tau} \right) =$$

$$= \left(\frac{dx^0}{d\tau}, \frac{dx^0}{d\tau}\frac{d\bar{x}}{dx^0} \right)$$

$$= \frac{dx^0}{d\tau}\left(1, \frac{\bar{v}}{c}\right)$$

to find $\frac{dx^0}{d\tau}$

$$\frac{c^2 \Delta\tau^2}{(\Delta x^0)^2} = \frac{(\Delta x^0)^2}{(\Delta x^0)^2} - \frac{(\Delta x)^2}{(\Delta x^0)^2}$$

taking the limit $(\Delta x^0) \to 0$

$$\left(c\frac{d\tau}{dt}\right)^2 = 1 - \frac{v^2}{c^2}$$

$$\frac{d\tau}{dx^0} = \frac{1}{c}\sqrt{1-\frac{v^2}{c^2}}$$

$$\boxed{\frac{dx^0}{d\tau} = \frac{c}{\sqrt{1-v^2/c^2}}}$$

This means

$$\boxed{\begin{aligned} \frac{dx^\mu}{d\tau} &= \frac{1}{\sqrt{1-v^2/c^2}}\,(c, \bar{v}) \\ p^\mu &= \frac{1}{\sqrt{1-v^2/c^2}}\,(mc, m\bar{v}) \end{aligned}}$$

The relativistic form of Newton's second law is

$$\frac{dp^\mu}{d\tau} = f^\mu$$

where $f^\mu = (0, \bar{F})$ in the particle's rest frame and transforms like a 4 vector.

In the absence of force

$$\frac{dp^\mu}{d\tau} = 0 \Rightarrow$$

p^μ is conserved.

$$\boxed{\begin{aligned} \gamma &= \frac{1}{\sqrt{1 - v^2/c^2}} \\ p^0 &= \gamma m c \\ \bar{p} &= \gamma m v \end{aligned}}$$

$p^0 c$ is identified with the energy and $\vec{p}$ is the conserved relativistic momentum $\neq m\vec{v}$

Principle of special relativity in quantum mechanics (Wigner 1939)

In an isolated system quantum observables cannot be used to distinguish different inertial reference frames

(this is different than the classical formulation which focuses on preserving the form of the equations in all inertial coordinate systems (as is the case with Maxwell's equations and the relativistic form of Newton's second law

The quantities that can be measured in quantum mechanics are

(1) probabilities $P_{AB} = |\langle A|B\rangle|^2$

(2) expectation values

$$\langle a|B|a\rangle = \sum |\langle a|b_n\rangle|^2 \, b_n$$

(3) ensemble averages

$$\mathrm{Tr}(\rho B) = \sum P_n^{a} \langle a_n|B|a_n\rangle$$

The only transformation that preserve these are unitary or antiunitary transformations

Consider

$$\begin{tikzcd} X_1 \arrow[rr, bend left, "\lambda_{12}\, a_{12}"] \arrow[r, no head, bend right, "\lambda_1 a_1"'] & X_2 \arrow[r, no head, bend right, "\lambda_2 a_2"'] & X_3 \end{tikzcd}$$

$$x_2 = \lambda_1 x_1 + a_1$$

$$x_3 = \lambda_2 x_2 + a_2$$

$$= \lambda_2(\lambda_1 x_1 + a_1) + a_2$$

$$= \lambda_2\lambda_1 x_1 + \lambda_2 a_1 + a_2$$

$$= \lambda_{12} x_1 + a_{12}$$

comparing these

$$\lambda_{12} = \lambda_2\lambda_1$$

$$a_{12} = \lambda_2 a_1 + a_2$$

this means

$$U(\lambda_2 a_2)\, U(\lambda_1 a_1) |\psi_1\rangle =$$

$$U(\lambda_2\lambda_1, \lambda_2 a_1 + a_2) |\psi_1\rangle$$

remarks

* each (λa) can be expressed as a product of squares of elementary transformations

$\therefore$ $U(\lambda a)$ cannot be antiunitary

* The states could differ by a phase

$$\mathcal{U}(\Lambda_2, a_2)\, \mathcal{U}(\Lambda_1, a_1) = \mathcal{U}(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2)\, e^{i\phi(2,1)}$$

It turns out for the Poincare group the phases can be eliminated by redefining the phases of the individual $\mathcal{U}(\Lambda a)$'s (up to the phases that arise in half integral spins in rotating by 2π)

$\boxed{\text{A relativistic quantum theory is defined by a unitary representation of the Poincare group acting on the Hilbert space}}$

$$\boxed{\begin{aligned} &\mathcal{U}(\Lambda_2 a_2)\,\mathcal{U}(\Lambda_1 a_1) = \mathcal{U}(\Lambda_2\Lambda_1,\ \Lambda_2 a_1 + a_2) \\ &\mathcal{U}^{+}(\Lambda a)\,\mathcal{U}(\Lambda a) = I \end{aligned}}$$

SL(2 C) It is useful to represent 4 vectors by 2x2 Hermitian matrices

$$\underline{\underline{X}} = \begin{pmatrix} x^0+x^3 & x^1-ix^2 \\ x^1+ix^2 & x^0-x^3 \end{pmatrix} = 2x^\mu \sigma_\mu$$

where

$$\sigma_\mu = (I, \bar{\sigma})$$

and $\bar{\sigma} = (\sigma_1\ \sigma_2\ \sigma_3)$ are Pauli matrices. Note

$$\underline{\underline{X}} = \underline{\underline{X}}^{+}$$

$$x^\mu = \frac{1}{2}\,\mathrm{Tr}\,(\sigma_\mu \underline{\underline{X}})$$

$$\det X = (x^0)^2 - \bar{x}\cdot\bar{x} = c^2 \tau^2$$

This means that any linear transformation that preserves

$$X = X^\dagger, \quad \det X$$

is a Lorentz transformation

$$X' = AXA^\dagger$$

with $\det A = 1$ satisfies these conditions since

$$(X')^\dagger = (A^\dagger)^\dagger X^\dagger A^\dagger = AXA^\dagger$$

$$\det X' = \det(A) \det X \det A^\dagger$$

$$= 1 \cdot \det X \cdot 1^*$$

$$= \det X$$

* while we could multiply A by $\alpha \neq 0$ and $A^\dagger$ by $1/\alpha$ this would not change the transformation

comments

(1) The group of 2x2 complex matrices with $\det A = 1$ is called $SL(2C)$

(2) A and $-A$ correspond to the same Lorentz transformati

(3) We can write $A = e^{B}$

$\det A = e^{\mathrm{Tr}\, B}$

a general matrix $B = \sum b^{\mu} \sigma_{\mu}$ where in the general case the b^{μ} are complex

$\mathrm{Tr}\, B = \sum b^{\mu}\, \mathrm{Tr}(\sigma_{\mu}) = 2 b^{0}$

$\det A = 1 = e^{2b^{0}}$

This means $b_0 = 0$ or $i\pi$

It follows that a general $SL(2C)$ matrix has the form

$$\boxed{A = \pm e^{\bar{b}\cdot\bar{\sigma}}}$$

where $\bar{b}$ is a complex 3 vector – properties

① $X' = AXA^{+} \Rightarrow$

$$\sum x'^{\mu}\sigma_{\mu} = \sum A\sigma_{\nu}A^{+}x^{\nu}$$

$$\tfrac{1}{2}\mathrm{Tr}(\sigma_{\alpha}x'^{\mu}\sigma_{\mu}) = \tfrac{1}{2}\mathrm{Tr}(\sigma_{\alpha}A\sigma_{\nu}A^{+}x^{\nu})$$

$$x'^{\alpha} = \tfrac{1}{2}\mathrm{Tr}(\sigma_{\alpha}A\sigma_{\nu}A^{+})x^{\nu}$$

$$\boxed{\Lambda^{\alpha}{}_{\nu} = \tfrac{1}{2}\mathrm{Tr}(\sigma_{\alpha}A\sigma_{\nu}A^{+})}$$

② if $\bar{b}$ is real

$A = A^{+}$

and

if $A = e^{\bar{r}\cdot\bar{\sigma}} = A^{\frac{\bar{r}}{2}\cdot\bar{\sigma}} A^{\frac{\bar{r}}{2}\cdot\bar{\sigma}}$

$= RR = R^{+}R \geq 0$

in this case A is a positive Hermitian matrix

③ if $\bar{b}$ is imaginary

$A = e^{i\bar{r}\cdot\bar{\sigma}}$ $\quad A^{+} = e^{-i\bar{r}\cdot\bar{\sigma}}$

$A^{+}A = I$

then A is unitary

④ $A(\lambda) = e^{\lambda\bar{b}\cdot\bar{\sigma}}$

* $A(\lambda) = A(\lambda/2)\,A(\lambda/2)$

* $A(\lambda) \to I$ continuously as $\lambda \to 0$

(this shows that any Lorentz transformation is a square.)

For Poincare transformation

$$\underline{\underline{X}} \to \underline{\underline{X}}' = \underline{\underline{A}}\,\underline{\underline{X}}\,\underline{\underline{A}}^{+} + \underline{\underline{B}}$$

where $\underline{\underline{B}}$ is a constant Hermitian matrix

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Construction of $U(\Lambda, a)$ for a single particle

Consider a particle like an electron. In order to characterize the state of an electron we need to know its linear momentum and the projection of its spin on a given axis

By changing frames
the 3 component of the
4 momentum can take on any value

$$p^\mu = \left(mc \frac{1}{\sqrt{1-v^2/c^2}}, m\bar{v} \frac{1}{\sqrt{1-v^2/c^2}}\right)$$

(note while $|v| < c$, $\frac{1}{\sqrt{1-v^2/c^2}}$ gets
large as $|v| \to c$)

Also the J_z can take
on 2 values

The Hilbert space for a
single electron is Hilbert

$\Psi_\mu(p)$

$$\sum_{\mu=-1/2}^{1/2} \int \Psi_\mu^*(\bar{p})\, \Psi_\mu(\bar{p})\, d^3p < \infty$$

we write these as $\langle \bar{p}\mu | \Psi \rangle$

$$\sum p^\mu p^\nu \eta_{\mu\nu} = \frac{m^2 v^2 - m^2 c^2}{1 - v^2/c^2} =$$

$$= m^2 c^2 \frac{v^2 - c^2}{c^2 - v^2} = -m^2 c^2$$

up to a factor of c

this is the rest mass of the electron. We write the basic state as

$$|(m\tfrac{1}{2})\,\bar{p}\,\mu\rangle$$

where $m, \frac{1}{2}$ are the fixed mass and spin of the electron

$\mathcal{U}(\Lambda, a) \rightarrow \mathcal{U}(R, 0)$ (rotation)

Consider

$$\mathcal{U}(R)\,|(m\tfrac{1}{2})\,\bar{0}\,\mu\rangle$$

* rotations do not change $\bar{0}$

$$\mathcal{U}(R)\,|(m\tfrac{1}{2})\,0\,\mu\rangle = \sum_{\nu=-\frac{1}{2}}^{\frac{1}{2}} |(m\tfrac{1}{2})\,0\,\nu\rangle\, \mathcal{M}_{\nu\mu}(R)$$

$M_{\mu\nu}(R)$ is a unitary 2×2 matrix representing a rotation in the basis of eigenstates of $J_z \rightarrow$ we know that this is just $M_{\mu\nu}(R) = D^{1/2}_{\mu\nu}(R)$

$$\boxed{\therefore\ \mathcal{U}(R,0)\,|(m\tfrac{1}{2})\,\bar{0}\,\mu\rangle = \sum_{\nu} |(m\tfrac{1}{2})\,0\,\bar{\nu}\rangle\, D^{\frac{1}{2}}_{\nu\mu}(R)}$$

Note – because rotations by 2π changes the sign of $D(R)$ it is useful to replace Lorentz transformation labels by SL(2C) labels

Homework

$$\Lambda^\mu{}_\nu = \frac{1}{2} \mathrm{Tr}\left(\sigma_\mu e^{\frac{\rho}{2}\sigma_z} \sigma_\nu e^{\frac{\rho}{2}\sigma_z}\right)$$

$$= \begin{pmatrix} \cosh\rho & 0 & 0 & \sinh\rho \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\rho & 0 & \cdot & \cosh\rho \end{pmatrix}$$

$$\Lambda^\mu{}_\nu \begin{pmatrix} mc \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} mc\cosh\rho \\ 0 \\ 0 \\ mc\sinh\rho \end{pmatrix}$$

these transformation change the 4 velocity

$$p^0 = mc\cosh\rho$$

$$\bar{p} = mc\sinh\rho\,\hat{p}$$

$$\boxed{\text{we call } A = e^{\frac{\bar{\rho}}{2}\cdot\bar{\sigma}} \text{ a rotationless Lorentz boost}}$$

It transforms $(mc\ 0\ 0\ 0)$

to $p^\mu = (\gamma mc, \gamma m\bar{v}) = (p^0, \bar{p})$

Let $B(\bar{p}/m) = e^{\bar{p}/2 \cdot \bar{\sigma}}$

We define $\mathcal{U}(B(p/m))$

$$\boxed{|(m\tfrac{1}{2})\,\bar{p}\,\mu\rangle \equiv \mathcal{U}(B(\tfrac{p}{m}))\,|(m\tfrac{1}{2})\,\bar{0}\,\mu\rangle\, N(\bar{p})}$$

* with this definition the spin projection does not change

* The normalization constant must be chosen so $\mathcal{U}(B(a))$ is unitary

To choose N note

$$\theta(p^0)\,\delta(m^2c^2+\bar{p}^2-(p^0)^2)\,d^4p =$$

$$\theta(p'^0)\,\delta(m^2c^2+\bar{p}'^2-(p'^0)^2)\,d^4p'$$

$$d^4p\,\frac{\delta(p^0-\sqrt{\bar{p}^2+m^2c^2})}{2\sqrt{\bar{p}^2+m^2c^2}} = \frac{\delta(p'^0-\sqrt{\bar{p}'^2+m^2c^2})}{2\sqrt{\bar{p}'^2+m^2c^2}}\,d^4p'$$

Integration over p^0, p'^0

$$\frac{d^3p}{2\sqrt{\bar{p}^2+m^2c^2}} = \frac{d^3p'}{2\sqrt{\bar{p}'^2+m^2c^2}}$$

define $\omega_m(\bar{p}) = \sqrt{m^2c^2+\bar{p}^2}$

$$\int |\bar{p}\rangle\, d^3p\, \langle\bar{p}| = I = u(\Lambda)\, I\, u^\dagger(\Lambda)$$

$$\int |\Lambda p\rangle N\, d^3p\, N\, \langle\Lambda p| \qquad \Lambda p = p'$$

$$\int |p'\rangle N^2 \frac{d^3p}{d^3p'}\, d^3p'\, \langle p'|$$

$$\boxed{N = \sqrt{\frac{d^3p'}{d^3p}} = \sqrt{\frac{\omega(\Lambda p)}{\omega(p)}}}$$

$$\boxed{u(\Lambda)|p\rangle = |\Lambda p\rangle\sqrt{\frac{\omega(\Lambda p)}{\omega(p)}}}$$

This means

$$\boxed{|(m\tfrac{1}{2})\,\bar{p}\,\mu\rangle = \mathcal{U}(B(p/m))\,|(m\tfrac{1}{2})\,\bar{0}\,\mu\rangle\sqrt{\frac{m}{\omega_m(p)}}}$$

This ensures unitarity.

Finally we consider translations of rest state

$$\mathcal{U}(I,a)\,|(m\tfrac{1}{2})\,0\,\mu\rangle =$$

$$e^{ip\cdot a/\hbar}\,|(m\tfrac{1}{2})\,0\,\mu\rangle =$$

$$e^{-ip^0a^0/\hbar}\,|(m\tfrac{1}{2})\,0\,\mu\rangle =$$

$$e^{-imca^0/\hbar}\,|(m\tfrac{1}{2})\,0\,\mu\rangle$$

$$\boxed{\mathcal{U}(I,a)\,|(m\tfrac{1}{2})\,0\,\mu\rangle = e^{-imca^0/\hbar}\,|(m\tfrac{1}{2})\,0\,\mu\rangle}$$

Note: The definition

$$U(B(\vec{p}/m), 0)\,|(m j)\,\vec{0}\,\mu\rangle = |(m j)\,\vec{p}\,\mu\rangle \sqrt{\frac{\omega(p)}{m}}$$

means that μ is the z component of the electron spin that would be measured in the electrons rest frame if we used $B^{-1}(\vec{p}/m)$ to transform to the rest frame

We can combine these three elementary transformation to construct a unitary representation of the Poincare group on the electrons Hilbert space

$$\mathcal{U}(\Lambda\, a)\,|(m j)\,\bar{p}\,\mu\rangle =$$

$$\mathcal{U}(I, a)\,\mathcal{U}(\Lambda, 0)\,|(m j)\,\bar{p}\,\mu\rangle =$$

$$\mathcal{U}(I\, a)\,\mathcal{U}(\Lambda, 0)\,\mathcal{U}(B(\tfrac{p}{m}), 0)\,|(m j)\,\bar{0}\,\mu\rangle \sqrt{\frac{m}{\omega(p)}} =$$

$$\mathcal{U}(I\, a)\,\mathcal{U}(B(\tfrac{\Lambda p}{m}), 0)\,\underline{\mathcal{U}(B^{-1}(\tfrac{\Lambda p}{m}), 0)} \times$$

$$\underline{\mathcal{U}(\Lambda, 0)\,\mathcal{U}(B(\tfrac{p}{m}), 0)}\,|(m j)\,0\,\mu\rangle \sqrt{\frac{m}{\omega(p)}}$$

$$\mathcal{U}(B(\tfrac{\Lambda p}{m}), 0)\,\mathcal{U}(I, B^{-1}(\tfrac{\Lambda p}{m})\, a) \times$$

$$\mathcal{U}(B^{-1}(\tfrac{\Lambda p}{m})\,\Lambda\, B(\tfrac{p}{m}), 0)\,|(m j)\,0\,\mu\rangle \sqrt{\frac{m}{\omega(p)}}$$

$$\mathcal{U}(B(\tfrac{\Lambda p}{m}), 0)\,\mathcal{U}(I, B^{-1}(\tfrac{\Lambda p}{m})\, a)$$

$$|(m j)\,0\,\nu\rangle\, D^{1/2}_{\nu\mu}\big(\underbrace{B^{-1}(\tfrac{\Lambda p}{m})\,\Lambda\, B(\tfrac{p}{m})}_{\text{rotation}}\big) \sqrt{\frac{m}{\omega(p)}}$$

rotation

$0 \to p \to \Lambda p \to 0$

(called Wigner rotation)

$$\mathcal{U}(B(\tfrac{\Lambda p}{m}), 0)\, e^{-i m c \cdot B^{-1}(\frac{\Lambda p}{m})\cdot a} \sum |(m j)\,0\,\nu\rangle$$

$$D^{\frac{1}{2}}_{\nu\mu}\big(B^{-1}(\tfrac{\Lambda p}{m})\,\Lambda\, B(p/m)\big) \sqrt{\frac{m}{\omega(p)}}$$

note

$$(m, \vec{0}) \cdot (B^{-1}(\Lambda p/m)\, a) =$$

$$(B(\Lambda p/m)(m, \vec{0})) \cdot (a)$$

$$(\Lambda p \cdot a)$$

so we get

$$\boxed{\begin{aligned} &\mathcal{U}(\Lambda, a)\, |(m\, \tfrac{1}{2})\, \bar{p}\, \mu\rangle = \\ &\sum e^{i \Lambda p \cdot a/m}\, |(m\, \tfrac{1}{2})\, \vec{\Lambda p}\, \nu\rangle \times \\ &D^{\frac{1}{2}}_{\nu\mu}\left(B^{-1}\left(\tfrac{\Lambda p}{m}\right) \Lambda\, B\left(\tfrac{p}{m}\right)\right) \sqrt{\frac{\omega(\Lambda p)}{\omega(p)}} \end{aligned}}$$

this is unitary since it involves products of unitary operators

This generalizes to arbitrary spins

$$\boxed{U(\Lambda,a)\,|(m\,j)\,\bar{p}\,\mu\rangle = \sum_\nu e^{i\Lambda p\cdot a}\,|(m\,j)\,\Lambda\bar{p}\,\nu\rangle \sqrt{\frac{\omega_m(\Lambda p)}{\omega_m(p)}} \times D^j_{\nu\mu}\left(B^{-1}\left(\frac{\Lambda p}{m}\right)\Lambda B\left(\frac{p}{m}\right)\right)}$$

If we take $U(\Lambda,a) = U(I, tc\ 000)$

$$U(t)\,|(m\,j)\,p\,\mu\rangle = e^{-ip^0ct/\hbar}\,|(m\,j)\,p\,\mu\rangle$$

where $p^0c = \gamma mc^2 = mc^2\frac{1}{\sqrt{1-v^2/c^2}}$

which is the relativistic energy

$$e^{-iEt/\hbar}$$

It follows tha

$$i\hbar \frac{d}{dt} \langle (mj)\, p\mu | U(t) | \psi \rangle$$

$$E \langle (mj)\, p\mu | U(t) | \psi \rangle$$

which is the ordinary Schrodinger equation with

$$E = p^0 c \quad ; \quad m^2c^2 = (p^0)^2 - (\bar{p})^2$$

$$E = c\sqrt{m^2c^2 + \bar{p}^2}$$

$$\boxed{\begin{aligned} &i\hbar \frac{d}{dt} \langle (mj)\, p\mu | U(t) | \psi \rangle \\ &\qquad c\sqrt{m^2c^2 + \bar{p}^2} \langle (mj)\, p\mu | U(t) | \psi \rangle \end{aligned}}$$

This is called the relativistic Schrodinger equation. It is a valid equation, but it gave the wrong magnetic moment using

$$p^\mu \to p^\mu - \frac{e}{c} A^\mu$$