Lecture 31

Relation between scattering operator and potential

Assume

(1) $AA^{+} = I$

(2) $\lim_{t \to \pm\infty} \| (I - A) e^{-iH_0 t/\hbar} |\psi\rangle \| = 0$

for both time limits

For $H' = A^{+} H A$

(1) $\lim_{t \to \pm\infty} e^{iH' t/\hbar} e^{-iH_0 t/\hbar} = A^{+} \Omega_{\pm}$

(2) $S' = \Omega_{+}'^{+} \Omega_{-}' = \Omega_{+}^{+} A A^{+} \Omega_{-} = \Omega_{+}^{+} \Omega_{-} = S$

proof

$$\lim_{t\to\pm\alpha} \| (e^{iH't/\hbar} e^{-iH_0t/\hbar} - A^+ e^{iHt/\hbar} e^{-iH_0t/\hbar})$$

$$\times |\psi\rangle\|$$

$$= \lim_{t\pm\alpha} \| A^+ (e^{iHt/\hbar} A e^{-iH_0t/\hbar} - e^{iHt/\hbar} e^{-iH_0t/\hbar})$$

$$|\psi\rangle\| =$$

$$= \lim_{t\to\pm\infty} \| A^+ e^{iHt/\hbar} (A - I) e^{-iH_0t/\hbar} |\psi\rangle\|$$

$$= \lim_{t\to\pm\infty} \| (A-I) e^{-iH_0t/\hbar} |\psi\rangle = 0$$

* remarks

$$|\psi_\pm'\rangle = \Omega_\pm' |\psi_\pm^0\rangle$$

$$= A^+ \Omega_\pm |\psi_\pm^0\rangle$$

$$= A^+ |\psi_\pm\rangle$$

this means that while

S does not change

the scattering wave functions change — but because phase shifts are a property of $S = e^{2i\delta}$, the phase shifts remain unchanged.

* we use the fact that

$$\lim_{t \to \pm\infty} \| (1-A) e^{-iH_0 t/\hbar} |\psi\rangle \| = 0$$

for <u>both</u> time limits in order to show

$$\Omega'_+ = A^\dagger \Omega_+ \quad \text{and} \quad \Omega'_- = A^\dagger \Omega_-$$

More interesting — this is a necessary and sufficient condition for H and H' to have the same S

To show the converse

(1) Assume $S' = S$ and

$$S' = \Omega_+'^{\dagger} \Omega_+' \qquad S = \Omega_+^{\dagger} \Omega_-$$

$$\Omega_+'^{\dagger} \Omega_-' = S' = S = \Omega_+^{\dagger} \Omega_- \Rightarrow$$

$$\boxed{\Omega_-' \Omega_-^{\dagger} = \Omega_+' \Omega_+^{\dagger}}$$

(2) note that $\Omega_+^{\dagger} |B\rangle = 0$ for bound states — define

$$A^{\dagger} = \Omega_{\pm}' \Omega_{\pm}^{\dagger} + \sum_n |b_n'\rangle\langle b_n|$$

then $A^{\dagger} A = A A^{\dagger} = I$

$$A^{\dagger} H = H' A^{\dagger}$$ or

$$H' = A^{\dagger} H A$$

but

$$A^{\dagger} \Omega_{\pm} = \Omega_{\pm}' \Omega_{\pm}^{\dagger} \Omega_{\pm} = \Omega_{\pm}'$$

$$0 = \lim \| (A^{+} e^{iHt/\hbar} e^{-iH_0 t/\hbar} - e^{iH't/\hbar} e^{-iH_0 t/\hbar}) |\psi\rangle \|$$

$$= \lim \| A^{+} e^{iHt/\hbar} (1 - A) e^{-iH_0 t/\hbar} |\psi\rangle \|$$

$$= \lim \| (1 - A) e^{-iH_0 t/\hbar} |\psi\rangle \|$$

This shows that $S = S'$

means that

$$H' = A^{+} H A$$

with

$$\lim_{t \to \pm\infty} \| (1 - A) e^{-iH_0 t/\hbar} |\psi\rangle \| = 0$$

* This condition depends on A but not on V.

* The fact that the wave function changes does not effect quantum observables

for ordinary operators

$$|\psi'\rangle = A^{+}|\psi\rangle$$

$$O' = A^{+} O A$$

means ordinary observables and ensemble averages are unchanged.

Relativity and Quantum Mechanics

* classically there are inertial coordinate systems where the laws of physics have the same form

In non relativistic classical mechanics different inertial coordinate systems are relates by translations, rotations, time translation and shifts by constant velocity

In relativistic classical mechanics the different inertial coordinate systems are related by translation, rotations, time shifts and velocity changing Lorentz transformation

Lorentz transformations preserve $c^2 \Delta t_{AB}^2 - \Delta x_{AB} \cdot \Delta x_{AB}$ between events where c is the speed of light in a vacuum.

* These have the property that the speed of light is the same in all inertial coordinate systems.

* Maxwells equations are preserved under Lorentz transformations

* Newtons Laws are preserved under Galilean transformations

These are not consistent.

Michelson Morley experiment determined that different inertial coordinate are related by Lorentz transformations

* principle of relativity –

experiments on isolated systems cannot distinguish different inertial coordinate systems

classically – the equations of motion have the same form in all inertial coordinate systems

In a quantum theory wave functions are not observable — Lorentz invariance in a quantum theory means that quantum measurements give identical results in all inertial coordinate systems.

The observables are probabilities expectation values and ensemble averages

These are left invariant by <u>unitary transformations</u>

<u>Poincaré group</u>

Consider 2 space time events A B with coordinates $(t_A \bar{x}_A)$ $(t_B \bar{x}_B)$

define

$$c^2 \Delta \tau_{AB} = c^2 (t_A - t_B)^2 - (\bar{X}_A - \bar{X}_B) \cdot (\bar{X}_A - \bar{X}_B)$$

where c is the speed of light in a vacuum. To give the space and time components identical units define

$$x^0 = ct$$

we also define a metric tensor

$$\eta_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with this notation

$$c^2 \Delta \tau_{AB}^2 = - \sum_{\mu\nu} \eta_{\mu\nu} (X_A - X_B)^\mu (X_A - X_B)^\nu$$

The Poincare group is the group of transformations that leave $c^2\Delta\tau^2$ invariant

* The most general transformation of the form

$$x^\mu \to x^{\mu'} = f^\mu(x)$$

that leaves $c^2\Delta\tau^2$ invariant is

$$\boxed{x^\mu \to x^{\mu'} = \sum \Lambda^\mu{}_\nu x^\nu + a^\mu}$$

where a^μ is a constant 4 component vector and $\Lambda^\mu{}_\nu$ is a constant matrix satisfying

$$\boxed{\sum_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \cdot \eta_{\mu\nu} = \eta_{\alpha\beta}}$$

These transformations are called Poincare transformations. The result can be proved by using

$$\sum_{\mu\nu} \eta_{\mu\nu} (X_A - X_B)^\mu (X_A - X_B)^\nu =$$

$$\sum \eta_{\mu\nu} (f^\mu(X_A) - f^\mu(X_B))(f^\nu(X_A) - f^\nu(X_B))$$

and differentiating with respect to X_A^μ X_B^ν and setting both to 0. (HW)

We can write these as matrix equations

$$X \to X' = \underline{\Lambda} X + a$$

$$\underline{\eta} = \underline{\Lambda}\, \underline{\eta}\, \underline{\Lambda}^T$$

The second equation $\Rightarrow$

$$\det \eta = \det(\Lambda \eta \Lambda^T) =$$

$$\det \Lambda \det \eta \det \Lambda^T =$$

$$\det \Lambda \det \eta \det \Lambda$$

which gives

$$\boxed{\det \Lambda^2 = 1}$$

We also have

$$\eta_{00} = \sum \Lambda^\mu{}_0 \Lambda^\nu{}_0 \eta_{\mu\nu}$$

$$= \Lambda^0{}_0 \Lambda^0{}_0 \eta_{00} + \sum \Lambda^i{}_0 \Lambda^i{}_0 \eta_{ii}$$

$$-1 = (\Lambda^0{}_0)^2 (-1) + \sum (\Lambda^i{}_0)^2$$

$$\boxed{(\Lambda^0{}_0)^2 = 1 + \sum (\Lambda^i{}_0)^2}$$

There are 4 classes of Lorentz transformation

Λ

$\det \Lambda = 1 \quad \Lambda^0_{\ 0} \geq 1 \quad \mathrm{I}$

$\det \Lambda = -1 \quad \Lambda^0_{\ 0} \geq 1 \quad \mathrm{II}$

$\det \Lambda = 1 \quad \Lambda^0_{\ 0} \leq -1 \quad \mathrm{III}$

$\det \Lambda = -1 \quad \Lambda^0_{\ 0} \leq -1 \quad \mathrm{IV}$

class II includes space reflections

class III includes both space and time reflections

class IV includes time reflection

The weak interaction is not invariant with respect to space reflections or time reversal.

The class of Lorentz transformations relevant for special relativity is class I ($\det \Lambda = 1$, $\Lambda^0_{\ 0} \geq 1$)

These transformations satisfy

$$x \to x' = \Lambda_1 x + a_1$$

$$x' \to x'' = \Lambda_2 (\Lambda_1 x + a_1) + a_2$$

$$= \underbrace{\Lambda_2 \Lambda_1}_{\Lambda_{21}} x + \underbrace{\Lambda_2 a_1 + a_2}_{a_{21}}$$

For $x \xrightarrow{\Lambda a} x'$ to preserve probabilities

$$|\psi'\rangle = U(\Lambda a)|\psi\rangle$$

for successive transformations

$$\begin{tikzcd} x \arrow[r, "\Lambda_1 a_1"'] \arrow[rr, bend left, "\Lambda_2\Lambda_1,\ \Lambda_2 a_1 + a_2"] & x' \arrow[r, "\Lambda_2 a_2"'] & x'' \end{tikzcd}$$

$$U(\Lambda_2 a_2)\, U(\Lambda_1 a_1) =$$

$$U(\Lambda_2 \Lambda_1, \Lambda_2 a_1 + a_2)\, e^{i\phi(12)}$$

for the Poincare group it is possible to redefine phases so $\phi(12) = 0$

Also while antiunitary transformations preserve probabilities, every transformation in I is a square, so the representation must be unitary.

Next consider

$$x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - i x^2 \\ x^1 + i x^2 & x^0 - x^3 \end{pmatrix} = X$$

note

(1) $X = X^\dagger$

(2) $\det X = (x^0)^2 - \vec{x}\cdot\vec{x} = c^2 \tau^2$

any transformation that preserves $X^\dagger = X$ and $\det X$ is a Lorentz transformation

$$X' = A X A^{+} \qquad \det A = 1$$

$$\det X' = \det(A X A^{+}) =$$

$$= \det(A) \det(X) \det(A^{+})$$

$$= 1 \cdot \det(X) \; 1 .$$

$$(X')^{+} = (A X A^{+})^{+} = A X^{+} A^{+} =$$

$$= A X A^{+} = X'$$

Using matrice

$$X' = A X A^{+} + \sigma_1$$

translation

Remark

$$\det A = 1 \Leftrightarrow A = e^{c^{\mu}\sigma_{\mu}}$$

$$\det A = e^{Tr(c^{\mu}\sigma_{\mu})} = e^{2c^{0}} = 1$$

$$c^{0} = 0, \pi \Rightarrow$$

$$\boxed{\begin{gathered} A = \pm e^{\vec{z}\cdot\vec{\sigma}} \\ \vec{z} = \text{complex vector} \end{gathered}}$$

Spin of particle of mass

m

$$p^\mu = m \frac{dx^\mu}{d\tau} = m \frac{dt}{d\tau} \cdot \frac{dx^\mu}{dt}$$

$$c^2 d\tau^2 = c^2 dt^2 - dx^2$$

$$c^2 = c^2 \left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dt}{d\tau}\right)^2 \left(\frac{dx}{dt}\right)^2$$

$$c^2 = \left(\frac{dt}{d\tau}\right)^2 (c^2 - v^2)$$

$$\left(\frac{dt}{d\tau}\right)^2 = \frac{1}{1 - v^2/c^2} = \gamma^2$$

$$\eta_{\mu\nu} p^\mu p^\nu = -m^2 \frac{c^2}{1 - v^2/c^2} + \frac{m^2 v^2}{1 - v^2/c^2}$$

$$= m^2 \left(\frac{v^2 - c^2}{1 - v^2/c^2} \right)$$

$$= -m^2 c^2$$

$$A = e^{-i\frac{\bar{\theta}\cdot\bar{\sigma}}{2}} \quad \text{Rotation}$$

$$A = e^{\frac{\rho}{2}\cdot\bar{\sigma}} \quad \text{Lorentz boost}$$

$$= \cosh\frac{\rho}{2}\, I + \hat{\rho} \sinh\frac{\rho}{2}$$

$$\cosh\rho = \frac{E}{mc^2} \quad \sinh\rho = \frac{p}{mc}$$

Note

$$A = \underbrace{(AA^{\dagger})^{1/2}}_{\text{positive (rotationless boost)}} \underbrace{(AA^{\dagger})^{-1/2} A}_{\text{unitary (rotation)}}$$

$$= \underbrace{A(A^{\dagger}A)^{-1/2}}_{\text{unitary (rotation)}} (A^{\dagger}A)^{1/2}$$

particles - state of a particle of mass m and spin j is given by its momentum and magnetic quantum # $-j \leq \mu \leq j$

denote this state by

$$|(mj)\,\bar{p}\,\mu\rangle$$

* when $\bar{p} = 0$ and $\Lambda = R$ (Rotation) $U(R)$ can only change μ

$$\mathcal{U}(R)\,|(mj)\,\bar{o}\,\mu\rangle =$$

$$\sum_{\nu} |(mj)\,\bar{o}\,\nu\rangle\langle j\nu|\,\mathcal{U}(R)\,|j\mu\rangle =$$

$$\sum_{\nu} |(mj)\,o\,\nu\rangle\, D^{j}_{\nu\mu}(R)$$

$$* \quad \boxed{\mathcal{U}(R)\,|(mj)\,\bar{o}\,\mu\rangle = \sum_{\nu=-j}^{j} |(mj)\,o\,\nu\rangle\, D^{j}_{\nu\mu}(R)}$$

For $\Lambda = B(P/m) =$

$$= \cosh\left(\frac{\rho}{2}\right) I + \hat{p}\cdot\bar{\sigma}\,\sinh\left(\frac{\rho}{2}\right)$$

where

$$\frac{E}{mc^2} = \cosh\rho \qquad \frac{P}{mc} = \sinh\rho$$

Applying $\mathcal{U}(B(P/m))$ to $|(mj)\,\bar{o}\,\mu\rangle$ changes the momentum from 0 to $\bar{P}$.

we define $|(mj)\,\bar{p}\,\mu\rangle$ by

$$|(mj)\,\bar{p}\,\mu\rangle =$$

$$N\,\mathcal{U}(B(p/m))\,|(mj)\,\bar{0}\,\mu\rangle$$

* The normalization constant is chosen to ensure $\mathcal{U}(B(p/m))$ is unitary

* The interpretation of μ is that it is the spin that would be measured in the rest frame if we transformed to the rest frame using $B^{-1}(p/m)$

For the choice of normalization

$$\langle (mj)\,\bar{p}'u' \,|\, (mj)\,\bar{p}u\rangle$$

$$= \delta(\bar{p}' - \bar{p})\,\delta_{u'u}$$

$$N = \sqrt{\omega_m(p)} \qquad \omega_m(p) = \sqrt{\bar{p}^2 + m^2c^2}$$

$$\boxed{\mathcal{U}(B(p/m))\,|(mj)\,\bar{0}\,u\rangle = |(mj)\,\bar{p}\,u\rangle \sqrt{\frac{\omega(p/m)}{mc}}}$$

* Finally for time evolution

$$\boxed{\mathcal{U}(I, a)\,|(mj)\,0\,u\rangle = e^{-i a^0 m c/\hbar}\,|(mj)\,0\,u\rangle}$$

We can use the 3 red boxed equations to get $\mathcal{U}(\Lambda\, a)$ for a particle of mass m and spin j.

$$\mathcal{U}(\Lambda\, a)\,|(m\,j)\,\bar{p}\,\mu\rangle =$$

$$\mathcal{U}(I\, a)\,\mathcal{U}(\Lambda, 0)\,|(m\,j)\,\bar{p}\,\mu\rangle$$

$$\mathcal{U}(I\, a)\,\mathcal{U}(\Lambda\, 0)\,\mathcal{U}(B(p), 0)\,|(m\,j)\,\bar{0}\,\mu\rangle \times \sqrt{\frac{m}{\omega(p)}} =$$

$$\mathcal{U}(I\, a)\,\mathcal{U}(B(\Lambda p/m))\,\mathcal{U}(B^{-1}(\Lambda p/m)) \times \mathcal{U}(\Lambda\, 0)\,\mathcal{U}(B(B(p), 0)\,|(m\,j)\,0\,\mu\rangle \sqrt{\frac{m}{\omega(p)}} =$$

$$\mathcal{U}(B(\Lambda p/m))\,\mathcal{U}(I, B^{-1}(\Lambda p/m)\,a) \times \mathcal{U}(\underbrace{B^{-1}(\Lambda p/m)\,\Lambda\, B(p/m)}_{\text{Rotati}})\,|(m\,j)\,0\,\mu\rangle \sqrt{\frac{m}{\omega(p)}} =$$

$$0 \to p \to \Lambda p \to 0$$

$$\mathcal{U}(B(\Lambda p/m))\,\mathcal{U}(I, B^{-1}(\Lambda p/m)\,a)\,|(m\,j)\,0\,\nu\rangle\, D^{j}_{\nu\mu}(B^{-1}(\Lambda p/m)\,\Lambda\, B(p/m))\,\sqrt{\frac{m}{\omega(p)}} =$$

$$\underbrace{e^{-i\, m c\, B^{-1}(\Lambda p/m)^{0}{}_{\mu}\, a}}_{e^{i\Lambda p\cdot a/\hbar}} \quad |(m\,j)\,0\,\nu\rangle$$

$$\boxed{\begin{aligned} U(\Lambda a)\,|(m j)\,\bar{p}\,\mu\rangle = \\ \sum_{\nu} |(m j)\,\overline{\Lambda p},\nu\rangle \times \\ D^{j}_{\nu\mu}\left(B^{-1}(\Lambda p)\Lambda B(p)\right)\sqrt{\frac{\omega(\Lambda p)}{\omega(p)}} \times \\ e^{i\Lambda p\cdot a/\hbar} \end{aligned}}$$

This gives an explicit unitary representation of the Poincaré group on the single particle Hilbert space