Lecture 30

Phase shifts

S unitary $\quad S = e^{2i\delta} \quad \delta = \delta^\dagger$

δ = phase shift operator

$$S = \sum Y_{em}(\hat{k}') \, S_e \, Y^*_{em}(\hat{k}) \qquad S_e = e^{2i\delta_e}$$

$$\langle \vec{k}' | T(E + i\epsilon) | k \rangle = \sum Y_{em}(\hat{k}') \, t_e(k) \, Y^*_{em}(\hat{k})$$

Last time

$$\boxed{-\pi \mu k \, t_e = e^{i\delta_e} \sin \delta_e}$$

This means that if you know δ_e for all e you can compute the differential cross section

Next I show the origin of the phase shift

For a finite range potential the partial wave scattering wave function has the following form for large r

$$\langle r \ell | k \ell_- \rangle = \frac{4\pi i^\ell}{(2\pi\hbar)^{3/2}} j_\ell\left(\frac{kr}{\hbar}\right)$$

$$\frac{(4\pi)^2}{(2\pi\hbar)^3}(-\pi\mu k) h_\ell^+\left(\frac{kr}{\hbar}\right) \int j_\ell\left(\frac{kr'}{\hbar}\right) V(r') r'^2 dr' \langle r' \ell | k \ell_- \rangle$$

In order to relate the right side to phase shifts

$$\langle \bar{k}' | T(E+i\epsilon) | \bar{k} \rangle =$$

$$\langle \bar{k}' | V | \bar{k}_- \rangle =$$

$$\int \langle \bar{k}' | \bar{r} \rangle d^3r \, V(r) \langle \bar{r} | \bar{k}_- \rangle =$$

$$\frac{4\pi}{(2\pi\hbar)^{3/2}}(-i)^\ell \int j_\ell\left(\frac{kr}{\hbar}\right) V(r)\, Y^*_{\ell m}(\hat{k})\, Y_{\ell m}(\hat{r})\, r^2 dr$$

$$\sum Y_{\ell' m'}(\hat{r}) \langle r \ell | k \ell_- \rangle Y_{\ell' m'}(\hat{k})$$

integrating over angles removes the spherical harmonics in r and gives kronecker δ functions

$$t_\ell = \frac{4\pi}{(2\pi\hbar)^{3/2}}(-i)^\ell \int j_\ell\left(\frac{kr}{\hbar}\right) V(r) \langle r \ell | k \ell_- \rangle r^2 dr$$

This means

$$\boxed{\int j_\ell\left(\frac{kr}{\hbar}\right) V(r)\, r^2 \langle r \ell | k \ell_- \rangle = \frac{(2\pi\hbar)^{3/2}}{4\pi}(i)^\ell t_\ell = \frac{(2\pi\hbar)^{3/2}}{4\pi}(i^\ell)\frac{1}{-\pi\mu k} \times e^{i\delta_\ell} \sin\delta_\ell}$$

$\langle r l | k l \rangle \to$

$$\frac{4\pi i^l}{(2\pi\hbar)^{3/2}} j_l\left(\frac{kr}{\hbar}\right) + \frac{(4\pi)^2}{(2\pi\hbar)^3}(-\pi\mu k)\, h_l^+\left(\frac{kr}{\hbar}\right)$$

$$\times \frac{(2\pi\hbar)^{3/2}}{4\pi} i^l \frac{1}{-\pi\mu k} e^{i\delta_l}\sin\delta_l =$$

$$\frac{4\pi i^l}{(2\pi\hbar)^{3/2}}\left[j_l\left(\frac{kr}{\hbar}\right) + h_l^+\left(\frac{kr}{\hbar}\right) e^{i\delta_l}\sin\delta_l\right]$$

for large r

$$j_l\left(\frac{kr}{\hbar}\right) \to \frac{\sin\left(\frac{kr}{\hbar} - \frac{l\pi}{2}\right)}{(kr/\hbar)}$$

$$h_l^+\left(\frac{kr}{\hbar}\right) \to \frac{e^{ikr/\hbar - i\frac{l\pi}{2}}}{(kr/\hbar)}$$

$$\frac{4\pi i^l}{(2\pi\hbar)^{3/2}} \frac{1}{2i}\left[\cancel{e^{i\left(\frac{kr}{\hbar} - \frac{l\pi}{2}\right)}} - e^{-i\left(\frac{kr}{\hbar} - \frac{l\pi}{2}\right)} \right.$$

$$\left. + e^{2i\delta_l + i\frac{kr}{\hbar} - \frac{l\pi}{2}} - \cancel{e^{i\frac{kr}{\hbar} - \frac{l\pi}{2}}}\right]$$

$$\frac{(4\pi) i^l}{(2\pi\hbar)^{3/2}} e^{i\delta_l}\left[\frac{e^{i\left(\frac{kr}{\hbar} - \frac{l\pi}{2} + \delta_l\right)} - e^{-i\left(\frac{kr}{\hbar} - \frac{l\pi}{2} + \delta_l\right)}}{2i}\right.$$

$$= \frac{4\pi i^{\ell}}{(2\pi\hbar)^{3/2}} e^{i\delta_\ell} \frac{\sin\left(\frac{kr}{\hbar} - \frac{\ell\pi}{2} + \delta_\ell\right)}{kr/\hbar}$$

except for the phase $e^{i\delta_\ell}$ this looks like the asymptotic form of a plane wave with phase shifted by δ.

Particles with spin. - more information about an interaction can be learned by considering the spin dependence of the scattering cross section

So far we have suppressed the spin dependence

$$\frac{d\sigma}{d\Omega}_{cm} = (2\pi)^4 \hbar^2 \mu^2 |\langle \bar{k}' \mu_1' \mu_2' \| T(E+i\epsilon) \| \bar{k} \mu_2 \mu_1 \rangle|^2$$

$$F = - \quad \hbar \quad \bar{k}' \mu_1' \mu'$$

$$d\sigma = |F|^2 d\Omega(\hat{k})$$

recall that in general the beam and target are from ensembles of many states. While they involve ensembles in momentum, if the beam or target is polarized then there are spin density matrices

$$\rho_T = \sum_i |u_{Ti}\rangle P_{Ti} \langle u_{Ti}|$$

$$\rho_B = \sum |u_{Bi}\rangle P_{Bi} \langle u_{Bi}|$$

It is customary to use the centercle mass cross section because it can be use to get to other f

$$\frac{d\sigma}{d\Omega} = \sum (2\pi)^4 \hbar^2 \mu^2 \times$$

$$\langle \bar{k}' \mu_1' \mu_2' \| T(E+i\epsilon) \| \bar{k} \mu_1 \mu_2 \rangle P_{T\mu_1} P_{B\mu_2}$$

$$\langle \bar{k} \mu_1 \mu_2 \| T(E-i\epsilon) \| k' \mu_1' \mu_2' \rangle$$

define

$$M_{\mu_1'\mu_2' \mu_1\mu_2} = -(2\pi)^2 \mu \hbar \langle k' \mu_1' \mu_2' \| T(E+i\epsilon) \| k \mu_1 \mu_2 \rangle$$

$$\frac{d\sigma}{d\Omega} = M \rho_B \rho_T M^+ =$$

$$M_{\mu_1'\mu_2'\mu_1\mu_2} P_{T\mu_1} P_{B\mu_2} M^+_{\mu_1\mu_2\mu_1'\mu_2'}$$

In this case if we do not measure polarization the cross section "

$$\frac{d\sigma}{d\Omega} = Tr\left(M \rho_B \rho_T M^{\dagger}\right)$$

as matrices in the spin degree of freedom

If O is a spin dependent quantity that we want to measure in the detectn then it can be to the unpolarized case (detect

$$\langle O \rangle = \frac{Tr\left(O M \rho_B \rho_T M^{\dagger}\right)}{Tr\left(M \rho_B \rho_T M^{\dagger}\right)}$$

Note that the most general

O is a matrix in spin

space

$$O = O_{\mu_1 \mu_2, \mu_1' \mu_2'}$$

The dimension of the matrix

is $(2j_1+1) \times (2j_2+1) \times$

$(2j_1+1) \times (2j_2+1) = N \times N$

a general $N \times N$ Hermitian

matrix can be expressed

in terms of a basis

of Hermitian matrices

$$\underline{\underline{O}} = \sum_{m=0}^{N^2-1} O_m \underline{\underline{S_m}}$$

where $S_m = S_m^{+}$ and the

S_m can be normalized

so $Tr(S_m) = N \delta_{m0}$

$T(S_m S_n) = N \delta_{mn}$

In this case

$$O S_m = \sum O_n S_n S_m$$

$$Tr(O S_m) = \sum O_n Tr(S_n S_m)$$

$$= N O_m$$

$$O_m = \frac{1}{N} Tr(O S_m)$$

In this case

$$\langle O \rangle = \frac{Tr(\sum O_m S_m M \rho_B \rho_T M^\dagger)}{Tr(M \rho_B \rho_T M^\dagger)}$$

$$= \sum O_m \frac{Tr(S_m M \rho_B \rho_T M^\dagger)}{Tr(M \rho_B \rho_T M^\dagger)}$$

This means that we can express the expectation value of any O if we have the N^2 $\langle S_m \rangle$

For spin $\frac{1}{2}$ particles the beam and target density matrices have the form

$$\rho = \frac{I + \vec{p}\cdot\vec{\sigma}}{2}$$

where $\vec{p}$ is called the polarization vector

Construction of S_i $(N \times N)$

* write down N^2 independent hermitian matrices H_i

* Let S_0 be the identity

  pick one of the remaining H

  $\tilde{S}_1 = H_1 - Tr(H S_0) S_0$

  normalize $Tr(\tilde{S}_1 \tilde{S}_1) = N$

  $\tilde{S}_2 = H_2 - Tr(H_2 S_0) S_0 - Tr(H_2 H_1) H_1$

then normilize to

$Tr(H_2 H_2) = N$

This is the same as using the Gram Schmid method on matrices with inner product $\langle M|N\rangle = Tr(M^{\dagger}N)$

Scattering of identical particles. - when particle are identical we cannot tell which one is detect

$$|F(\vec{k}'\vec{k})|^2 \rightarrow$$

$$|F(\vec{k}'\vec{k}) \pm F(-\vec{k}',\vec{k})|^2$$

where the + sign is for integer spin identical particles and − is for half integer spin particles the purpose of scattering is to try to learn about the microscopic interactions between particles.

Can the potential be determined?

The answer is No

To show this let

(1) $H = H_0 + V$

(2) Let A satisfy

$AA^{+} = I$

$\lim_{t \to \mp\infty} \| (I - A) e^{-iH_0 t/\hbar} |\psi\rangle \| = 0$ (both time limits)

(3) $H' = A^{+} H A$

$\Downarrow$

H and H' give the same S operator

$$\lim_{t\to\pm\infty} e^{iH't/\hbar} e^{-iH_0 t/\hbar} |\psi\rangle =$$

$$\lim_{t\to\pm\infty} A^{+} e^{iHt} (A - I + I) e^{-iH_0 t/\hbar} |\psi\rangle =$$

$$\lim_{t\to\pm\infty} A^{+} e^{iHt/\hbar} e^{-iH_0 t/\hbar} |\psi\rangle +$$

$$\lim_{t\to\pm\infty} \underbrace{A^{+} e^{iHt/\hbar}}_{\text{unitary}} (A - I) e^{-iH_0 t/\hbar} |\psi\rangle$$

$$\Downarrow$$

$$\| A^{+} e^{iHt/\hbar} (A-I) e^{-iH_0 t/\hbar} |\psi\rangle \| =$$

$$\| (A-I) e^{-iH_0 t/\hbar} |\psi\rangle \|$$

(This vanishes by assumption as $t \to \pm\infty$

$$\therefore \quad \Omega_{\pm}(H' H_0) = A^{+} \Omega_{\pm}(H H_0)$$

$$S' = \Omega_{+}'^{+} \Omega_{-}' = \Omega_{+}^{+} \underbrace{A A^{+}}_{I} \Omega_{-} = \Omega_{+}^{+} \Omega_{-} = S$$

We remark that the converse also holds - assume $S = S'$

This means

$$\Omega_+^\dagger \Omega_- = \Omega_+'^\dagger \Omega_-' \implies$$

$$\Omega_+ \Omega_+'^\dagger = \Omega_- \Omega_-'^\dagger \equiv A^\dagger$$

$$\Omega_+ \Omega_+'^\dagger H' = \Omega_+ H_0 \Omega_+'^\dagger = H \Omega_+ \Omega_+'^\dagger$$

similarly

$$\Omega_- \Omega_-'^\dagger H' = \Omega_- H_0 \Omega_-'^\dagger = H \Omega_- \Omega_-'^\dagger$$

This means

$$A^\dagger H' = H A^\dagger \quad \text{or}$$

$$A^\dagger H' A = H$$

$$\Omega_+ = \lim e^{iHt/\hbar} e^{-iH_0 t/\hbar}$$

$$= \lim e^{iA^\dagger H' A t/\hbar} e^{-iH_0 t/\hbar}$$

$$= \boxed{\lim A^\dagger e^{iH't/\hbar} A e^{-iH_0 t/\hbar}}$$

$$= A^\dagger \Omega_+' =$$

$$= \boxed{\lim A' e^{iH't/\hbar} e^{-iH_0 t/\hbar}}$$

Taking the difference give

$$0 = \lim_{t \to \pm\infty} \| e^{iH't/\hbar} A^{+} (1-A) e^{-iH_0 t/\hbar} |t\rangle \|$$

$$= \lim_{t \to \pm\infty} \| (1-A) e^{-iH_0 t/\hbar} |t\rangle \| = 0$$

* Note that A has nothing to do with the potential

* It is important that

$$\Omega_+ \Omega_+^{+} = \Omega_- \Omega_-^{+}$$

* what this means that even a complete measurement of all independed scattering variables are not sufficient to recover a unique potential

operators A satisfying the condition $AA^{\dagger} = 1$ and

$$\lim_{t \to \pm\infty} \| (A - I) e^{-iH_0 t/\hbar} |\psi\rangle \| = 0$$

$$A = \frac{1 + iP}{1 - iP} \qquad P = \sum_{k=1}^{N} |k\rangle n_k \langle k|$$

where $\langle k_i | k_j \rangle = \delta_{ij}$ — clearly there are a large class of unitary operators that leave S unchanged.