

Lecture 2

Last time

$$SU(2) \quad X = \vec{x} \cdot \vec{\sigma} \quad \vec{x} = \frac{1}{2} \text{Tr}(\vec{\sigma} X)$$

$$\det(X) = -|\vec{x}|^2$$

$$X = X^\dagger \quad (\vec{x}^* = \vec{x})$$

$$\text{Tr}(X) = 0$$

conditions preserved

$$X \rightarrow X' = UXU^\dagger \quad (\text{rotations})$$

$$UU^\dagger = I \quad \det(U) = 1$$

$$R_{ij} = \frac{1}{2} \text{Tr}(\sigma_i U \sigma_j U^\dagger)$$

homework

$$U = e^{-\frac{i}{2} \vec{\theta} \cdot \vec{\sigma}}$$

$$= \cos\left(\frac{\theta}{2}\right) I - i \hat{\theta} \cdot \vec{\sigma} \sin\left(\frac{\theta}{2}\right)$$

$$= e_0 + i \vec{e} \cdot \vec{\sigma} \quad e_0^2 + \vec{e}^2 = 1$$

space reflection

$$X \rightarrow -X$$

cannot be expressed as UXU^\dagger

$$U, -U$$

give the same R_{ij}

2 component spinors

$$\chi \begin{pmatrix} a \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\chi' = U \chi \quad \text{rotates spinors}$$

General case $U(R) = e^{-i \vec{J} \cdot \vec{\theta} / \hbar} \quad \hbar = 1$

$$[J_i, V_j] = i \sum_{k=1}^3 \epsilon_{ijk} V_k$$

(transformation of vector operators under rotation)

\vec{J} conserved vector

$$[J_i, J_j] = i \sum_{k=1}^3 \epsilon_{ijk} J_k$$

(means \vec{J} transforms like a vector)

$$\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$$

Homework: show angular momentum commutation relations imply

$$[\vec{J}^2, J_i] = 0 \quad \text{any } i$$

This means we can find simultaneous eigenstates of \vec{J}^2 and J_z $|n, u\rangle$

$$J^2 |n, u\rangle = n |n, u\rangle$$

$$J_z |n, u\rangle = u |n, u\rangle$$

properties

$$J_{\pm} = J_x \pm iJ_y$$

Homework show commutation relations imply

$$[J_z, J_{\pm}] = \pm J_{\pm}$$

$$J_z J_{\pm} = J_{\pm} (J_z \pm 1)$$

$$J_z \underline{J_{\pm} |n, u\rangle} = \underline{(u \pm 1) J_{\pm} |n, u\rangle}$$

this means either

$$J_z J_{\pm} |n, u\rangle = 0$$

or

$$J_{\pm} |n, u\rangle = |n, u \pm 1\rangle C$$

where C is a normalization constant

normalization

Homework Commutation relation.

$$J_{\mp} J_{\pm} = \bar{J}^2 - J_z^2 \mp J_z$$

$$\|J_{\pm}|m\rangle\|^2 =$$

$$\langle m|J_{\pm}^{\dagger} J_{\pm}|m\rangle =$$

$$\langle m|J_{\mp} J_{\pm}|m\rangle =$$

$$\langle m| \bar{J}^2 - J_z^2 \mp J_z |m\rangle$$

$$(m - m(m \pm 1))$$

* since this is a norm it must be non negative

* as $m \rightarrow \pm \infty$ this eventually becomes negative

this means $J_{\pm}|m\rangle = 0$ to the lowest and highest possible values of m

$$m = m_{\max}(m_{\max} + 1) = m_{\min}(m_{\min} - 1)$$

this gives a quadratic equation relating u_{\min} to u_{\max}

$$u_{\min}^2 - u_{\min} - u_{\max}(u_{\max} + 1) = 0$$

$$u_{\min} = \frac{1}{2} \left(1 \pm \sqrt{1^2 + 4u_{\max}^2 + 4u_{\max}} \right)$$

$$= \frac{1}{2} \left(1 \pm \sqrt{(2u_{\max} + 1)^2} \right)$$

$$= \frac{1}{2} \begin{cases} 2 + 2u_{\max} & = u_{\max} + 1 \\ -2u_{\max} & = -u_{\max} \end{cases}$$

since $u_{\min} \leq u_{\max}$ $u_{\min} \neq u_{\max} + 1$

$$\therefore u_{\min} = -u_{\max}$$

$$u_{\max} - u_{\min} = \text{integer}$$

$$2u_{\max} = \text{integer}$$

$$u_{\max} = \text{integer} / 2$$

standard convention is to define

$$j = u_{\max}$$

replace $n = j(j+1)$ by j

$$|ju\rangle \equiv |nu\rangle$$

$$\vec{J}^2 |j, m\rangle = j(j+1) |j, m\rangle$$

$$J_z |j, m\rangle = m |j, m\rangle$$

$$J_{\pm} |j, m\rangle = |j, m \pm 1\rangle \sqrt{j(j+1) - m(m \pm 1)}$$

$$= |j, m \pm 1\rangle \sqrt{(j \mp m)(j \pm m + 1)}$$

$$j^2 - m^2 + j \mp m$$

$$j(j+1) - m(m \pm 1)$$

Wigner Function

$$e^{-i \vec{J} \cdot \vec{\theta}} |j, v\rangle$$

x since $[\vec{J}, J^2] = 0$ rotation

$$J^2 e^{-i \vec{J} \cdot \vec{\theta}} |j, v\rangle =$$

$$e^{-i \vec{J} \cdot \vec{\theta}} J^2 |j, v\rangle = j(j+1) e^{-i \vec{J} \cdot \vec{\theta}} |j, v\rangle$$

$$\langle j, m | \mathcal{U}(\vec{\theta}) | j', m' \rangle = \delta_{jj'} D_{mv}^j(R(\vec{\theta}))$$

$\underbrace{\hspace{1cm}}_{\text{SU}(2)}$

$$\langle j, m | \mathcal{U}(R_2) \mathcal{U}(R_1) | j', m' \rangle =$$

$$\sum \langle j, m | \mathcal{U}(R_2) | j'', m'' \rangle \langle j'', m'' | \mathcal{U}(R_1) | j', m' \rangle$$

$$D_{mv}^j(R_2) D_{v'm'}^{j'}(R_1) = D_{mv}^j(R_2 R_1)$$

$$U^\dagger(R) U(R) = I$$

$$\langle f'u | U^\dagger(R) | f'u' \rangle \langle f'u' | U(R) | f''u'' \rangle =$$

$$D_{u'u}^{f'}(R)^* D_{u''u'}^f(R) = \delta_{ff'} \delta_{u''u}$$

$$D_{uu'}^f(R)^\dagger D_{u''u'}^f(R)$$

this means $D_{uv}^f(R)$ is a $2f+1$ dimensional representation of $SU(2)$

these are needed in many applications - even in computer games

method of Schwinger

$$n_\pm = f \pm u$$

$$f = \frac{1}{2}(n_+ + n_-)$$

$$u = \frac{1}{2}(n_+ - n_-)$$

definitions - we can use n_\pm

$$\text{as labels for } |f'u\rangle = | \frac{n_+ + n_-}{2}, \frac{n_+ - n_-}{2} \rangle$$

$$:= |n_+ n_- \rangle$$

In this notation

$$J_{\pm} |j, m\rangle = |j, m \pm 1\rangle \sqrt{j(j+1) \mp m(m \pm 1)}$$

becomes

$$J_+ |n_+, n_-\rangle = |n_+, n_- - 1\rangle \sqrt{n_-(n_- + 1)}$$

$$J_- |n_+, n_-\rangle = |n_+, n_- + 1\rangle \sqrt{n_+(n_+ + 1)}$$

Schwinger defines non Hermitian operators

$$a_{\pm} |0, 0\rangle = 0$$

$$a_+ |n_+, n_-\rangle = |n_+, n_- - 1\rangle \sqrt{n_-}$$

$$a_- |n_+, n_-\rangle = |n_+, n_- + 1\rangle \sqrt{n_+}$$

$$a_+^{\dagger} |n_+, n_-\rangle = |n_+, n_- + 1\rangle \sqrt{n_+ + 1}$$

$$a_-^{\dagger} |n_+, n_-\rangle = |n_+, n_- - 1\rangle \sqrt{n_- + 1}$$

exercise show these definitions

imply

$$[a_+, a_+^{\dagger}] = [a_-, a_-^{\dagger}] = 1$$

and all other commutators vanish and

$$N_{\pm} = a_{\pm}^{\dagger} a_{\pm}$$

$$N_{\pm} |n_+, n_-\rangle = n_{\pm} |n_+, n_-\rangle$$

From these definitions

$$\begin{aligned} J_+ |n_+ n_- \rangle &= |n_+ + 1, n_- \rangle \sqrt{n_- (n_- + 1)} \\ &= a_+^\dagger a_- |n_+ n_- \rangle \end{aligned}$$

$$\begin{aligned} J_- |n_+ n_- \rangle &= |n_+, n_- + 1 \rangle \sqrt{n_+ (n_+ + 1)} \\ &= a_-^\dagger a_+ |n_+ n_- \rangle \end{aligned}$$

we also have

$$\begin{aligned} J_z |n_+ n_- \rangle &= \frac{1}{2} (n_+ - n_-) |n_+ n_- \rangle \\ &= \frac{1}{2} (a_+^\dagger a_+ - a_-^\dagger a_-) |n_+ n_- \rangle \end{aligned}$$

using the definitions

$$J_\pm = J_x \pm i J_y$$

gives (homework)

$$J_x = \frac{1}{2} (a_+^\dagger a_- + a_-^\dagger a_+) = \frac{1}{2} \mathbf{a}^\dagger \sigma_x \mathbf{a}$$

$$J_y = \frac{i}{2} (a_-^\dagger a_+ - a_+^\dagger a_-) = \frac{1}{2} \mathbf{a}^\dagger \sigma_y \mathbf{a}$$

putting everything's together gives

$$\boxed{\vec{J} = \frac{1}{2} (a_+^\dagger \ a_-^\dagger) \vec{\sigma} \begin{pmatrix} a_+ \\ a_- \end{pmatrix} = \frac{1}{2} \mathbf{a}^\dagger \vec{\sigma} \mathbf{a}}$$

We use this to calculate

$$D_{uv}^{\dagger}(R)$$

$$|n_+ n_-\rangle = \frac{(a_+^{\dagger})^{n_+}}{n_+!} \frac{(a_-^{\dagger})^{n_-}}{n_-!} |00\rangle$$

follows from

$$a_+^{\dagger} |n_+\rangle = |n_++1\rangle \sqrt{n_++1}$$

$$|n_++1\rangle = \frac{1}{\sqrt{n_++1}} \cdot a_+^{\dagger} |n_+\rangle$$

$$|n_+\rangle = \frac{a_+^{\dagger}}{\sqrt{n_+!}} |n_+-1\rangle$$

use induction to show the result

$$\langle n_+ n_- | = \langle 0 | \frac{(a_+)^{n_+}}{n_+!} \frac{(a_-)^{n_-}}{n_-!}$$

$$U(R) = e^{-i\vec{\theta} \cdot \vec{T}} = e^{-i\frac{1}{2}\vec{\theta} \cdot (a^{\dagger} \vec{\sigma} a)}$$

$$D_{uv}^{\dagger}(R) = \langle 00 | \frac{(a_+)^{n_+}}{\sqrt{n_+!}} \frac{(a_-)^{n_-}}{\sqrt{n_-!}} \underbrace{e^{-i\frac{\vec{\theta}}{2} \cdot (a^{\dagger} \vec{\sigma} a)}}_U \frac{(a_+^{\dagger})^{n_+}}{\sqrt{n_+!}} \frac{(a_-^{\dagger})^{n_-}}{\sqrt{n_-!}} |00\rangle$$

We start by writing ? a_i^{\dagger}

$$\frac{u(a_+^{\dagger})U^{\dagger} u(a_+^{\dagger})U \cdots u(a_-^{\dagger})U^{\dagger} u(a_-^{\dagger})U \cdots U |00\rangle}{\sqrt{n_+!} \sqrt{n_-!}}$$

$$D_{uv}^{\dagger}(R) = \langle 00 | \frac{(a_+^{\dagger})^{n_+}}{\sqrt{n_+!}} \frac{(a_-^{\dagger})^{n_-}}{\sqrt{n_-!}} \frac{(u a_+^{\dagger} u^{\dagger})^{n_+}}{\sqrt{n_+!}} \frac{(u a_-^{\dagger} u^{\dagger})^{n_-}}{\sqrt{n_-!}} u | 00 \rangle$$

note 1

$$u | 00 \rangle = (I - i \frac{\bar{\theta}}{2} \cdot a^{\dagger} \sigma a + \dots) | 00 \rangle = | 00 \rangle$$

since $a | 00 \rangle = 0$ on all terms except I have an a on the right

note 2

$$X(\lambda) = e^{\lambda A} B e^{-\lambda A}$$

$$\frac{dX}{d\lambda} = e^{\lambda A} [A, B] e^{-\lambda A}$$

$$\frac{d^n X}{d\lambda^n} = e^{-\lambda A} [A [A \dots B]] e^{-\lambda A}$$

$$X(\lambda) = B + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \frac{d^n}{d\lambda^n} (X(\lambda)) \Big|_{\lambda=0}$$

$$= B + \lambda [A, B] + \frac{\lambda^2}{2!} [A [A, B]] + \dots$$

set $\lambda = 1$

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!} [A [A, B]] + \frac{1}{3!} [A [A [A, B]]] + \dots$$

In this case $A = -\frac{i}{2} a^\dagger \bar{\psi} \cdot \bar{\psi} a$ $B = a_\pm^\dagger$

$$\begin{aligned} [A, B] &= -\frac{i}{2} \sum [a_i^\dagger (\bar{\psi} \cdot \bar{\psi})_{i\pm}, a_\pm] \\ &= -\frac{i}{2} \sum a_i^\dagger (\bar{\psi} \cdot \bar{\psi})_{i\pm} \end{aligned}$$

repeated

$$\begin{aligned} [A, [A, B]] &= \\ &= -\frac{i}{2} \sum a_i^\dagger (\bar{\psi} \cdot \bar{\psi})_{i\pm} a_\pm, -\frac{i}{2} \sum a_k^\dagger (\bar{\psi} \cdot \bar{\psi})_{k\pm} \\ &= \left(-\frac{i}{2}\right)^2 a_i^\dagger (\bar{\psi} \cdot \bar{\psi})_{i\pm}^2 \end{aligned}$$

continuing the n^{th} commutator gives

$$\rightarrow \left(-\frac{i}{2}\right)^n a_i^\dagger (\bar{\psi} \cdot \bar{\psi})_{i\pm}^n$$

the full exponential series gives

$$\begin{aligned} (U a_\pm^\dagger U^\dagger) &= \sum a_i^\dagger \left(\sum \frac{1}{n!} \left(-\frac{i}{2}\right)^n (\bar{\psi} \cdot \bar{\psi})_{i\pm}^n \right) \\ &= \sum a_i^\dagger \left(e^{-\frac{i}{2} \bar{\psi} \cdot \bar{\psi}} \right)_{i\pm} \end{aligned}$$

$$= \sum a_i^\dagger R_{i\pm} = (a_+^\dagger R_{++} + a_-^\dagger R_{-+})$$

where

$$\underline{R} = \cos\left(\frac{\theta}{2}\right)I - i \vec{\sigma} \cdot \hat{\theta} \sin\left(\frac{\theta}{2}\right)$$

returning to the expression for D_{uv}^\dagger

$$D_{uv}^\dagger(R) = \langle 00 | \frac{(a_+^\dagger)^{n_+}}{\sqrt{n_+!}} \frac{(a_-^\dagger)^{n_-}}{\sqrt{n_-!}} \frac{(a_+^\dagger R_{++} + a_-^\dagger R_{-+})^{n_+}}{\sqrt{n_+!} \sqrt{n_-!}} (a_+^\dagger R_{+-} + a_-^\dagger R_{--})^{n_-} | 00 \rangle$$

next we use the binomial series

$$(a+b)^n = \sum_{k=0}^n \frac{n!}{k!(n-k)!} a^k b^{n-k}$$

$$D_{uv}^\dagger(R) = \langle 00 | \frac{(a_+^\dagger)^{n_+}}{\sqrt{n_+!}} \frac{(a_-^\dagger)^{n_-}}{\sqrt{n_-!}} \frac{n_+! n_-!}{\sqrt{n_+!} \sqrt{n_-!}} \times$$

$$\sum_{m=0}^{n_+} \sum_{k=0}^{n_-} \frac{(a_+^\dagger R_{++})^m}{m!} \frac{(a_-^\dagger R_{-+})^{n_+-m}}{(n_+-m)!} \frac{(a_+^\dagger R_{+-})^k}{k!} \frac{(a_-^\dagger R_{--})^{n_--k}}{(n_--k)!} | 00 \rangle$$

the important observation is that this vanishes unless the number of a_\pm^\dagger = the number of a_\pm — also not

$$\langle 0 | (a_\pm)^n (a_\pm^\dagger)^n | 0 \rangle = n!$$

$$D_{uv}^{\dagger}(R) = \sum_{m=1}^{n_+} \sum_{k=1}^{n_-} \frac{n_+! n_-!}{\sqrt{n_+! n_-! n_+! n_-!}} \frac{n_+! n_-!}{m! (n_+ - m)! k! (n_- - k)!}$$

$$R_{++}^m R_{-+}^{n_+ - m} R_{+-}^k R_{--}^{n_- - k}$$

$$= \sum \sum \sqrt{n_+! n_-! n_+! n_-!} \frac{R_{++}^m}{m!} \frac{R_{-+}^{n_+ - m}}{(n_+ - m)!} \frac{R_{+-}^k}{k!} \frac{R_{--}^{n_- - k}}{(n_- - k)!}$$

The last step is to convert $n_+ n_-$ back to $f u v$

$$n_+^{\dagger} = f + u \quad n_+ = f + v$$

$$n_-^{\dagger} = f - u \quad n_- = f - v$$

$$n_+ - m = f + v - m$$

$$n_- - k = f - v - k$$

we also have

$$m + k = n_+^{\dagger}$$

$$n_+ - m + n_- - k = n_-^{\dagger}$$

we can eliminate $k = n_+^{\dagger} - m = f + u - m$

$$\sum_{m=1}^{f+v} \frac{\sqrt{(f+u)! (f-u)! (f+v)! (f-v)!}}{m! (f+v-m)! (f+u-m)! (f-v-f-u+m)!}$$

$$R_{++}^m R_{-+}^{f+v-m} R_{+-}^{f+u-m} R_{--}^{m-v-u}$$

we finally get

$$D_{uv}^j(R) = \sum_{m=1}^{j+v} \frac{(j+u)!(j-u)!(j+v)!(j-v)!}{m!(j+v-m)!(j+u-m)!(m-u-v)!} \\ R_{++}^m R_{-+}^{j+v-m} R_{+-}^{j+u-m} R_{--}^{m-v-u}$$

Exercise

show $D_{uv}^{\frac{1}{2}} = R_{uv}$