Lecture 29

Last time

$$\langle \bar r | \vec k_- \rangle = \langle \bar r | \bar k \rangle + \int \underbrace{\langle \bar r | (\mathcal{E} - h_0 + i\varepsilon) | \bar r' \rangle}\, V(r')\, d^3 r' \,\langle \bar r' | \bar k_- \rangle$$

$$\langle \bar r | \bar k \rangle = \sum_{lm} \frac{(4\pi)\, i^l}{(2\pi\hbar)^{3/2}} j_l\left(\frac{kr}{\hbar}\right) Y_{lm}(\hat r)\, Y^*_{lm}(\hat k)$$

(Rotationally invariant V)

$$\langle \bar r | \bar k_- \rangle = \sum_{lm} \frac{(4\pi)\, i^l}{(2\pi\hbar)^{3/2}} j_l\left(\frac{kr}{\hbar}\right) Y_{lm}(\hat r)\, Y^*_{lm}(\hat k) + \sum_{lm} \frac{(4\pi)^2}{(2\pi\hbar)^3} \boxed{\left(-\pi \sim k \int j_l\left(\frac{kr_<}{\hbar}\right) h^+\left(\frac{kr_>}{\hbar}\right)\right)} V(r')\, r' \, d^3 r' \, Y_{lm}(\hat r)\, Y^*_{lm}(\hat r')\langle \bar r' | \bar k_- \rangle$$

where $j_l\left(\frac{kr}{\hbar}\right) = \frac{1}{2i}\left(h_l^+\left(\frac{kr}{\hbar}\right) - h_l^-\left(\frac{kr}{\hbar}\right)\right)$

Define

$$\boxed{\begin{aligned} |k^- l m\rangle &= \int |\bar k^-\rangle\, d\Omega(\hat k)\, Y_{lm}(\hat k) \\ |\bar k_-\rangle &= \sum_{lm} |k_- l\rangle\, Y^*_{lm}(\hat k) \end{aligned}}$$

$$|r\ell m\rangle = \int |\vec{r}\rangle\, Y_{\ell m}(\hat{r})\, d\Omega(\hat{r})$$

$$|\vec{r}\rangle = \sum_{\ell m} |r\ell m\rangle\, Y^{*}_{\ell m}(\hat{r})$$

using these definitions the Lippmann Schwinger equation becomes

$$\sum_{\ell m} Y_{\ell m}(\hat{r}) \langle r\ell m | k\_\ell' m'\rangle\, Y_{\ell' m'}(\hat{k}) =$$

$$\sum_{\ell m} 4\pi\, i^{\ell} j_\ell\left(\frac{kr}{\hbar}\right) Y_{\ell m}(\hat{r})\, Y_{\ell m'}(\hat{k}) \cdot$$

$$\int \frac{(4\pi)^2}{(2\pi\hbar)^3}(-\pi\mu k) \sum_{\substack{\ell m \\ \ell' m'}} \int j_\ell\left(\frac{kr_<}{\hbar}\right) h^{+}_{\ell}\left(\frac{kr_>}{\hbar}\right) V(r') \times$$

$$Y_{\ell m}(\hat{r})\, Y^{*}_{\ell m}(\hat{r}')\, d^3r' \times$$

$$Y_{\ell' m'}(\hat{r}') \langle r'\ell' m' | k\_\ell'' m''\rangle\, Y'_{\ell'' m''}(\hat{k})$$

integrating over $\hat{r}'$ and multiplying by $Y^{*}_{\ell''' m'''}(\hat{r}),\ Y_{\ell''' m'''}(\hat{k})$ and integrating over $\hat{k}$ and $\hat{r}$ removes all of the spherical harmonics

$$\langle r\ell m | k^- \ell m\rangle = \frac{4\pi i^\ell j_\ell(kr/\hbar)}{(2\pi\hbar)^{3/2}} - \frac{(4\pi)^2\, \pi \mu k}{(2\pi\hbar)^3}\int j_\ell\left(\frac{kr_<}{\hbar}\right) h^+\left(\frac{kr_>}{\hbar}\right) V(r')\, r'^2 \langle r'\ell m | k^- \ell m\rangle$$

c that the equation is the same for all m values

so we replace $|r\ell m\rangle \to |r\ell\rangle$

$|k_-\ell m\rangle \to |k_-\ell\rangle$

The equation in the red box is the partial wave Lippmann Schwinger equation

* solve the equation for each partial wave

$\Rightarrow |k^-\ell\rangle$

* construct the full solution

$$|\vec{k}_-\rangle = \sum_{m\ell} |k^-\ell\rangle Y_\ell^m(\hat{k})$$

* calculate

$\langle \bar{k} \| V \| \bar{k}_- \rangle$

* $d\sigma = \frac{(2\pi)^4 \hbar^3}{V} |\langle \bar{k}' \| V \| \bar{k}_- \rangle|^2$

$\delta(E' - E)\, \delta(\bar{p}' - \bar{p})\, d^3 p_1'\, d^3 p_2'$

example δ shell potential

$V(r) = -V_0 R_0 \delta(r - R_0)$

(the R_0 outside is so V_0 has dimensions of energy

$$\langle r | k_- \ell \rangle = \sqrt{\frac{2}{\pi}} \frac{1}{\hbar^{3/2}}\, i^\ell j_\ell(kr/\hbar)$$

$$+ \frac{2\mu k}{\hbar^3} \int_0^\infty j_\ell\left(\frac{kr_<}{\hbar}\right) h_\ell\left(\frac{kr_>}{\hbar}\right) V(r')\, r'^2 \langle r' | k_- \ell \rangle$$

In this example the integral gives

$$j_\ell\left(\frac{kr_<}{\hbar}\right) h_\ell\left(\frac{kr_>}{\hbar}\right)(-V_0 R_0^3)\langle R_0 | k_- \ell \rangle$$

where

$$j_\ell\left(\frac{kr_<}{\hbar}\right) = \begin{cases} j_\ell\left(\frac{kR}{\hbar}\right) & r>R \\ j_\ell\left(\frac{kr}{\hbar}\right) & r<R \end{cases}$$

$$h_\ell^+\left(\frac{kr_>}{\hbar}\right) = \begin{cases} h_\ell^+\left(\frac{kR}{\hbar}\right) & R>r \\ j_\ell\left(\frac{kr}{\hbar}\right) & r>R \end{cases}$$

The right side of the equation only depends on $\langle R_0\ell|k\ell\rangle$ which is a single number. Setting $r = R_0$ gives a linear equation for $\langle R_0\ell|k\ell\rangle$

$$\langle R_0\ell|k\ell\rangle = \sqrt{\frac{2}{\pi}}\frac{1}{(\hbar)^{3/2}}\, i^\ell\, j_\ell\left(\frac{kR_0}{\hbar}\right) - \frac{2\mu k}{\hbar^3}(-V_0R_0)R_0^2\, j_\ell(kR_0/\hbar)\, h_\ell^+\left(\frac{kR_0}{\hbar}\right) \times \langle R_0\ell|k\ell\rangle$$

solving for $\langle R_0\ell|k\ell\rangle$ gives

$$\boxed{\langle R_0 \ell | k\_\ell \rangle = \frac{\sqrt{\frac{2}{\pi}} \frac{i^{\ell}}{(\hbar)^{3/2}} j_\ell\left(\frac{kR_0}{\hbar}\right)}{1 + \frac{2\mu k V_0 R_0^3}{\hbar^3} j_\ell\left(\frac{kR_0}{\hbar}\right) h_\ell^{+}\left(\frac{kR}{\hbar}\right)}}$$

once we know this we can substitute it into the equation on the integral equation

$$\boxed{\begin{aligned} \langle r \ell | k\_\ell \rangle = {} & \sqrt{\frac{2}{\pi}}\, i^{\ell} \frac{1}{\hbar^{3/2}} j_\ell\left(\frac{kr}{\hbar}\right) \\ & (-)^{2} \frac{\pi k \mu R_0^3 V_0}{\hbar^3} \times \frac{\sqrt{\frac{2}{\pi}} \frac{i^{\ell}}{\hbar^{3/2}} j_\ell\left(\frac{kR_0}{\hbar}\right)}{1 + \frac{2\mu k V_0 R_0^3}{\hbar^3} j_\ell\left(\frac{kR_0}{\hbar}\right) h_\ell^{+}\left(\frac{kR_0}{\hbar}\right)} \times \\ & j_\ell\left(\frac{kr_<}{\hbar}\right) h_\ell^{+}\left(\frac{kr_>}{\hbar}\right) \end{aligned}}$$

where for $r > R$

$$j_\ell\left(\frac{kr_<}{\hbar}\right) h_\ell^{+}\left(\frac{kr_>}{\hbar}\right) = j_\ell\left(\frac{kR}{\hbar}\right) h_\ell^{+}\left(\frac{kr}{\hbar}\right)$$

next we want to express the solution of this equation in terms of phase shifts

Def: Phase shifts

* Any unitary operator U can be expressed as e^{iH} where $H = H^{+}$

* since S is unitary it can be expressed as $S = e^{2i\delta}$ where $\delta = \delta^{+}$ is a Hermitian operator called the phase shift operator

$\langle \bar{P}' k' \ell' m' | S | \bar{P}' k' \ell' m' \rangle =$

$$\delta(P-P') \frac{\delta(k-k')}{k^2} \delta_{\ell\ell'} \delta_{mm'} e^{2i\delta_\ell}$$

since H conserves energy

since

$$\langle k | S | k' \rangle = \langle k' | k \rangle - 2\pi i \delta(E-E') \langle k | T | | k' \rangle$$

it is useful to replace $|k|$ by E

$$I = \sum_{\ell m} |k \ell m\rangle k^2 dk \langle k \ell m|$$

$$E = k^2/2\mu \qquad dE = \frac{2k\,dk}{2\mu} = \frac{k}{\mu} dk$$

$$k = \sqrt{2\mu E} \qquad dk = \frac{\mu}{k} dE = \frac{\mu}{\sqrt{2\mu E}} dE$$

$$k^2 = 2\mu E$$

$$I = \sum |k \ell m\rangle k^2 \frac{\mu}{k} dE \langle k \ell m|$$

$$= \sum |E \ell m\rangle dE \langle E \ell m|$$

$$\Rightarrow \boxed{|E \ell m\rangle = |k \ell m\rangle \sqrt{k\mu}}$$

In the energy basis

$$\langle E \ell m | S | E' \ell' m' \rangle =$$

$$S_\ell(E)\, \delta(E-E')\, \delta_{\ell\ell'}\, \delta_{mm'} =$$

$$\delta(E-E')\, \delta_{\ell\ell'}\, \delta_{mm'} \left(1 - 2\pi i \langle E \ell m | T | E \ell m \rangle\right)$$

$$\delta(E-E')\, \delta_{\ell\ell'}\, \delta_{mm'} \left(1 - 2\pi i k\mu \langle k \ell m | T | k \ell m \rangle\right)$$

since energy and angular momentum are conserved

$$\langle k \ell m | T | k' \ell' m' \rangle = t_\ell(k)$$

$$\boxed{S_\ell(E) = e^{2i\delta_\ell} = 1 - 2\pi i k\mu\, t_\ell(k)}$$

$$-2\pi i k\mu\, t_\ell = e^{2i\delta_\ell} - 1$$

$$= e^{i\delta_\ell}\left(e^{i\delta_\ell} - e^{-i\delta_\ell}\right)$$

$$-\pi k\mu\, t_\ell = e^{i\delta_\ell}\, \frac{e^{i\delta_\ell} - e^{-i\delta_\ell}}{2i} =$$

$$\boxed{-\pi \mu k\, t_\ell = e^{i\delta_\ell} \sin\delta_\ell}$$

If we define f_e by

$$F = -(2\pi)^2 \mu \hbar \; T$$

$$\sum Y_{em}(\hat{k}) f_e(k) Y^{*}_{em}(\hat{k}') =$$

$$-(2\pi)^2 \mu \hbar \sum Y_{em}(\hat{k}) t_e(k) Y^{*}_{em}(\hat{k}')$$

$$f_e(k) = -(2\pi)^2 \mu \hbar \, t_e$$

$$-\pi \mu k \, t_e = e^{i\delta_e} \sin \delta_e =$$

$$-\pi \mu k \left(\frac{-f_e}{4\pi^2 \mu \hbar} \right) = \frac{k}{4\pi \hbar} f$$

$$\boxed{f_e = \frac{4\pi \hbar}{k} e^{i\delta_e} \sin \delta_e}$$

We can use this to ress

the scattering ampli

in terms of the ph

shifts

$$\frac{d\sigma}{d\Omega(k)} = |F(k' k)|^2 =$$

$$\left| \sum_{\ell m} Y_{\ell m}(\hat{k}') f_\ell(k) Y^*_{\ell m}(\hat{k}) \right|^2 =$$

$$\left| \sum_{\ell m} Y_{\ell m}(\hat{k}') \frac{4\pi\hbar}{k} e^{i\delta_\ell} \sin\delta_\ell Y^*_{\ell m}(\hat{k}) \right|^2$$

integrating over $\hat{k}'$ gives

$$\sigma_T = \sum_{\ell m} \left(\frac{4\pi\hbar}{k}\right)^2 \sin^2\delta_\ell \underbrace{Y_{\ell m}(\hat{k}) Y_{\ell m}(\hat{k})}$$

$$\frac{2\ell+1}{4\pi} P_\ell(\hat{k}\cdot\hat{k})$$

$$\frac{2\ell+1}{4\pi} P_\ell(1)$$

$$\frac{2\ell+1}{4\pi}$$

$$\boxed{\sigma_T = \sum_\ell \frac{4\pi\hbar^2}{k^2} (2\ell+1) \sin^2\delta_\ell}$$

next we return to the integral equation

$$\langle r\ell | k\_\ell \rangle = \frac{4\pi i^\ell}{(2\pi\hbar)^{3/2}} j_\ell\left(\frac{kr}{\hbar}\right)$$

$$- \frac{4\pi^2 \pi \mu k}{(2\pi\hbar)^3} j_\ell\left(\frac{kr_<}{\hbar}\right) h_\ell^+\left(\frac{kr_>}{\hbar}\right) r'^2 V(r')\, dr' \langle r'\ell | k\_\ell \rangle$$

note

$$\langle \bar{k}' \| T(E+i\epsilon) \| \bar{k} \rangle = \langle \bar{k}' \| V \| \bar{k} \rangle =$$

$$\langle k' \| \bar{r} \rangle V(r)\, d^3r \langle \bar{r} | \bar{k} \rangle =$$

$$\int \sum \frac{4\pi (-i)^\ell}{(2\pi\hbar)^{3/2}} Y_{\ell m}(\hat{k}') Y_{\ell m}^*(\hat{r}) j_\ell\left(\frac{k'r}{\hbar}\right) V(r') \times$$

$$\sum Y_{\ell' m'}(\hat{r}) \langle r\ell | k\ell \rangle Y_{\ell m}^*(k)$$

$$\boxed{\frac{(4\pi)(-i)^\ell}{(2\pi\hbar)^{3/2}} j_\ell\left(\frac{kr}{\hbar}\right) V(r) r^2 dr \langle r\ell | k\ell\_ \rangle}$$

$$\times Y_{\ell m}(\hat{k}') Y_{\ell m}^*(\hat{k})$$

$$= \sum \boxed{t_\ell(k)} Y_{\ell m}(\hat{k}') Y_{\ell m}^*(k)$$

the expression for $r >$ range of potential becomes

$$\langle r\ell | k\_\ell \rangle = \frac{(4\pi) i^\ell}{(2\pi\hbar)^{3/2}} j_\ell\left(\frac{kr}{\hbar}\right) -$$

$$\underbrace{\frac{(4\pi)^2 \pi \mu \hbar}{(2\pi\hbar)^3} \frac{(2\pi\hbar)^{3/2}}{4\pi} i^\ell}_{\frac{4\pi i^\ell}{(2\pi\hbar)^{3/2}}} t_\ell(k)\, h_\ell^+\left(\frac{kr}{\hbar}\right)$$

$$= \frac{4\pi i^\ell}{(2\pi\hbar)^{3/2}}\left[j_\ell\left(\frac{kr}{\hbar}\right) - i\mu\hbar\, \underbrace{t_\ell(k)}_{-e^{i\delta_\ell}\sin\delta_\ell}\, h_\ell^+\left(\frac{kr}{\hbar}\right)\right]$$

$$\frac{4\pi i^\ell}{(2\pi\hbar)^{3/2}}\left[j_\ell\left(\frac{kr}{\hbar}\right) + e^{i\delta_\ell}\sin\delta_\ell\, h_\ell^+\left(\frac{kr}{\hbar}\right)\right]$$

next we consider the asymptotic forms of $j_\ell\left(\frac{kr}{\hbar}\right)$ and $h_\ell^+\left(\frac{kr}{\hbar}\right)$

$$j_\ell(x) \to \frac{1}{x}\sin\left(x - \frac{\ell\pi}{2}\right)$$

$$h_\ell^+(x) \to \frac{1}{x} e^{i\left(x - \frac{\ell\pi}{2}\right)}$$

This means that for large r the wave function behaves like

$$\frac{4\pi i^\ell}{(2\pi\hbar)^{3/2}} \times \frac{\hbar}{kr}\left(\sin\left(\frac{kr}{\hbar}-\frac{\ell\pi}{2}\right) + e^{i\left(\frac{kr}{\hbar}-\frac{\ell\pi}{2}\right)+i\delta_\ell}\sin\delta_\ell\right)$$

$$\frac{4\pi i^\ell \hbar}{(2\pi\hbar)^{3/2} kr}\left[\frac{\cancel{e^{i\frac{kr}{\hbar}-i\frac{\ell\pi}{2}}} - e^{-i\frac{kr}{\hbar}+i\frac{\ell\pi}{2}}}{2i} + \frac{e^{i\frac{kr}{\hbar}-i\frac{\ell\pi}{2}+2i\delta} - \cancel{e^{i\frac{kr}{\hbar}-\frac{\ell\pi}{2}}}}{2i}\right.$$

$$\frac{2\pi i^{\ell-1}\hbar}{(2\pi\hbar)^{3/2}}\frac{1}{kr}\left[e^{i\delta_\ell}\left(e^{i\delta_\ell+i\frac{kr}{\hbar}-i\frac{\ell\pi}{2}} - e^{-i\delta_\ell-i\frac{kr}{\hbar}+i\frac{\ell\pi}{2}}\right)\right.$$

$$\frac{4\pi i^\ell\hbar}{(2\pi\hbar)^{3/2}}\frac{1}{kr}e^{i\delta_\ell}\sin\left(\frac{kr}{\hbar}-\frac{\ell\pi}{2}+\delta_\ell\right)$$

while $j_\ell\left(\frac{kr}{\hbar}\right) \to \dfrac{\sin\left(\frac{kr}{\hbar}-\frac{\ell\pi}{2}\right)}{kr}$

This shows that the potential causes the phase of the incoming wave to shift by δ_ℓ

This is why $S = e^{2i\delta}$ with

δ called the phase shift