Lecture 27

$$d\sigma = \frac{(2\pi)^4}{V} \hbar^2 |\langle \bar{p}_1' \bar{p}_2' || V || (p_1 p_2)_- \rangle|^2 \times$$

$$\delta(\bar{P}' - \bar{P})\, \delta(E' - E)\, d^3p_1'\, d^3p_2'$$

where

$$|(p_1 p_2)_-\rangle = |p_1 p_2\rangle +$$

$$(E - H_0 + i\epsilon)^{-1} V |(p_1 p_2)_-\rangle$$

consider the coordinate space version

$$\bar{P} = \bar{p}_1 + \bar{p}_2 \qquad k = \frac{m_2 p_1 - m_1 p_2}{m_1 + m_2}$$

$$\bar{R} = \frac{m_1 \bar{r}_1 + m_2 r_2}{m_1 + m_2} \qquad r = |r_1 - r_2|$$

$$E = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} = \frac{P^2}{2M} + \frac{k^2}{2\mu}$$

$$|(\bar{P}\bar{k})_-\rangle = |\bar{P}\bar{k}\rangle +$$

$$\left(\cancel{\frac{P^2}{2M}} + \frac{k^2}{2\mu} - \cancel{\frac{P^2}{2M}} - \frac{\hat{k}^2}{2\mu} + i\epsilon\right) V |(\bar{P}\bar{k})_-\rangle$$

$$\Rightarrow |(Pk)_-\rangle = |P\rangle \times |(R)_-\rangle$$

provided V commutes with $\bar{P}$. The relevant integral equation is for $|(k)_-\rangle$ $\left(H_0 = h_0 + \frac{P^2}{2\mu} \right)$

$$\langle \bar{r} | (\bar{k})_- \rangle = \langle \bar{r} | \bar{k} \rangle + \int d^3r' \langle \bar{r} | \left(\frac{k^2}{2\mu} - h_0 + i\epsilon \right)^{-1} | \bar{r}' \rangle V(r') \langle \bar{r}' | k_- \rangle$$

The first step is to compute

$$\langle \bar{r} | \left(\frac{k^2}{2\mu} - h_0 + i\epsilon \right)^{-1} | \bar{r}' \rangle =$$

$$\frac{1}{(2\pi\hbar)^{3/2}} \int e^{i\bar{k}' \cdot \bar{r}/\hbar} \frac{d^3k'}{\frac{k^2}{2\mu} - \frac{k'^2}{2\mu} + i\epsilon} \frac{1}{(2\pi\hbar)^{3/2}} e^{-i\bar{k}' \cdot \bar{r}'/\hbar} =$$

$$\frac{2\mu}{(2\pi\hbar)^3} \int \frac{e^{i\bar{k}' \cdot (\bar{r} - \bar{r}')} d^3k'}{k^2 - k'^2 + 2\mu\epsilon i}$$

To do this integral choose coordinates so $\bar{r} - \bar{r}'$ is in the z direction

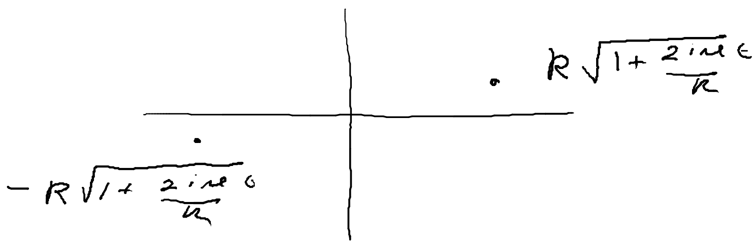
$$\frac{2\mu}{(2\pi\hbar)^3}\int \frac{e^{ik'|r-r'|\cos\theta}}{k^2-k'^2+2\mu i\epsilon}\, k'^2\, dk'\, \sin\theta\, d\theta\, d\phi$$

note $k^2 - k'^2 + 2i\mu\epsilon =$

$$-\left(k' - \sqrt{k^2+2i\mu\epsilon}\right)\left(k' + \sqrt{k^2+2i\mu\epsilon}\right)$$

$$= -\left(k' - k\sqrt{1+\frac{2i\mu\epsilon}{k}}\right)\left(k' + k\sqrt{1+\frac{2i\mu\epsilon}{k}}\right)$$

In the complex plane there are zeroes at



$*$ with this choice of coordinates the ϕ integral gives 2π

* to do the θ integral

let $u = \cos\theta$ $\quad du = -\sin\theta\, d\theta$

the limits of integration

become $0 \to \pi \to 1 \to -1$

$$\int_0^{\pi} \sin\theta\, d\theta = \int_{-1}^{1} du$$

$$= -\frac{2\mu(2\pi)}{(2\pi\hbar)^3} \int_0^{\infty} k'^2 dk' \int_{-1}^{1} du \frac{e^{i|r-r'|k' u/\hbar}}{(k' - k\sqrt{1+\Delta})(k' + k\sqrt{1+\Delta})}$$

$$\Delta = \frac{2i\mu\epsilon}{k}$$

doing the u integral

gives

$$= -\frac{2\mu \cdot 2\pi}{(2\pi\hbar)^3} \int_0^{\infty} k'^{\cancel{2}} dk' \left[\frac{1}{(k' - k\sqrt{1+\Delta})(k' + k\sqrt{1+\Delta})} \right]$$

$$\frac{\hbar}{i|r-r'|\cancel{k'}} \left(e^{i|r-r'|k/\hbar} - e^{-i|r-r'|k/\hbar} \right)$$

In the integral involving

the second exponent let

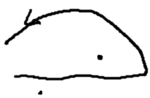
$k'' = -k'$ $\qquad k' dk' = k'' dk''$,

$[0 \to \infty] \to [0, -\infty]$ $\qquad k'^2 - k^2 \to k''^2 - k^2$

This allows us to extend the integral $\int_0^\infty \to \int_{-\infty}^{\infty}$

$$\frac{i\,2u\hbar}{(2\pi\hbar)^3}\,\frac{2\pi}{|\bar r-\bar r'|}\int_{-\infty}^{\infty}\frac{k''\,dk''\,e^{ik''|r-r'|/\hbar}}{(k''-k\sqrt{1+\Delta})(k''+k\sqrt{1+\Delta})}$$

The integral can be computed using the residue theorem.



closing the contour in the upper half plane means there is no contribution from the semicircle in the limit that the radius $\to \infty$

$$\oint f(z)\,dz = (2\pi i)\sum \mathrm{Res}(z_i)$$

where $f(z) \sim \dfrac{\mathrm{Res}(z)}{z-z_0}$

$$i\,\frac{2u\hbar}{(2\pi\hbar)^3}\,\frac{(2\pi)^2 i}{|r-r'|}\,\frac{k}{2k}\,e^{ik|r-r'|/\hbar} =$$

$$-\frac{u}{2\pi\hbar^2}\,\frac{e^{ik|r-r'|}}{|r-r'|}$$

with this the integral equation for $\langle \bar{r} | (k)_- \rangle$ becomes

$$\boxed{\langle \bar{r} | (\bar{k})_- \rangle = \langle \bar{r} | \bar{k} \rangle - \frac{\mu}{2\pi\hbar^2} \int \frac{e^{ik|r-r'|}}{|r-r'|} V(r') \, d^3r' \langle \bar{r}' | (k)_- \rangle}$$

note

$$|r-r'| = \sqrt{r^2 + r'^2 - 2\bar{r}\cdot\bar{r}'}$$

The potential $V(r')$ cuts off the r' integral, so for $r \gg r'$ this can be approximated by

$$|r-r'| = r\sqrt{1 - 2\frac{\bar{r}\cdot\bar{r}'}{r^2} + \frac{r'^2}{r^2}}$$

recall for $\Delta \ll 1$

$$\sqrt{1+\Delta} = 1 + \frac{1}{2}\Delta - \frac{1}{8}\Delta^2 \cdots$$

$$|r-r'| = r\left(1 + \frac{1}{2}\left(-\frac{2\bar{r}\cdot\bar{r}'}{r^2} + \frac{r'^2}{r^2}\right) - \frac{1}{8}\left(-\frac{2\bar{r}\cdot\bar{r}'}{r^2} + \frac{r'^2}{r^2}\right)^2 + \cdots\right)$$

ignoring terms that fall off

$$\to r - \frac{\bar{r}'\cdot\hat{r}}{r} + \frac{r'^2}{r} - \frac{1}{2}\frac{(\bar{r}\cdot\bar{r}')^2}{r^3} + \cdots$$

$$\frac{1}{|r-r'|} = \frac{1}{r}\left(1 - \frac{\bar{r}'\cdot\bar{r}}{r^2} + \cdots\right)$$

$$e^{ik|r-r'|/\hbar} = e^{ikr\left(1 - \frac{\bar{r}'\cdot\bar{r}}{r^2} + \cdots\right)/\hbar} \simeq e^{ikr/\hbar}\, e^{-ik\bar{r}'\cdot\hat{r}/\hbar} \cdots$$

The wave function gets squared in the cross section.

$$\langle \vec{r} | k \rangle_- \rightarrow \langle \vec{r} | \vec{k} \rangle - \frac{\mu}{2\pi\hbar^2} \frac{e^{ikr/\hbar}}{r} \int e^{\frac{-ik\hat{r}\cdot\vec{r}'}{\hbar}} V(r') \langle r' | k \rangle_-$$

+ corrections that vanish faster than $\frac{1}{r}$ for large

It is useful to define

$$\vec{k}' \equiv k\hat{r}$$

which has the energy of the initial state, in the direct $\hat{r}$. Then we write

$$-\frac{\mu}{2\pi\hbar^2}(2\pi\hbar)^{3/2}\underbrace{\int\frac{1}{(2\pi\hbar)^{3/2}}e^{-i\bar{k}\cdot\bar{r}'/\hbar}\,V(r')\,T\,|\bar{k}\rangle}_{\langle\bar{k}'|T|k\rangle}$$

we finally get

$$\langle r|(\bar{k})\rangle \longrightarrow \frac{1}{(2\pi\hbar)^{3/2}}\Big[e^{i\bar{k}\cdot\bar{r}/\hbar}$$

$$-\underbrace{\frac{\mu}{2\pi\hbar^2}(2\pi\hbar)^3}_{(2\pi)^2\mu\hbar}\langle\bar{k}'|T|k\rangle\frac{e^{ikr/\hbar}}{r}\Big]$$

This means that the coefficient of the outgoing spherical wave is

$$\boxed{-(2\pi)^2\mu\hbar\,\langle\bar{k}'|T(E+i\epsilon)|\bar{k}\rangle}$$

This is exactly the scattering amplitude defined earlier

$$\boxed{\begin{aligned} F(\vec{k}',\vec{k}) &= -(2\pi)^2 \mu \hbar \langle \vec{k}' \| T(E+i\epsilon) \| \vec{k} \rangle \\ \left(\frac{d\sigma}{d\Omega}\right)_{cm} &= |F|^2 \end{aligned}}$$

* This is where the (ϵ) sign introduced earlier appeared.

The optical theorem

The unitarity of the scattering operator has some useful computational implications

$$\langle p_1' p_2' | S^\dagger S | p_1 p_2 \rangle = \langle p_1' p_2' | p_1 p_2 \rangle =$$

$$\langle \bar{p}_1' \bar{p}_2' | p_1 p_2 \rangle$$

$$+ 2\pi i\, \delta(E'-E)\, \delta(\bar{p}'-p) \langle p_1' p_2' \| T^\dagger \| p_1 p_2 \rangle$$

$$- 2\pi i\, \delta(E-E')\, \delta(\bar{p}-p') \langle p_1' p_2' \| T \| p_1 p_2 \rangle$$

$$+$$

$$+\int (2\pi)^2\, \delta(E-E'')\,\delta(P-P'')\,\delta(E'-E'')\,\delta(\bar{P}'-P'')$$

$$\langle p_1' p_2'| T(E-i\epsilon) |p_1'' p_2''\rangle\, dp_1''\, dp_2''$$

$$\langle p_1'' p_2''| T(E+i\epsilon) |p_1 p_2\rangle$$

replace

$$\delta(E-E'')\,\delta(\bar{P}-\bar{P}'')\,\delta(P''-\bar{P}')\,\delta(E''-E')$$

$$d^3p_1''\, d^3p_2'' \quad =$$

$$\delta(E-E')\,\delta(\bar{P}-\bar{P}') \times$$

$$\delta(E-E'')\,\delta(P-P'')\, d^3p_1'' d^3p_2''$$

and set $\bar{p}_1' = p_1$ $\quad p_2' = \bar{p}_2$ to get

$$0 = -2\pi i \langle p_1 p_2| \big(T(E+i\epsilon) - T(E-i\epsilon)\big) |p_1 p_2\rangle$$

$$+ (2\pi)^2 \int |\langle p_1'' p_2''| T(E+i\epsilon) |p_1 p_2\rangle|^2$$

$$\delta(E-E'')\,\delta(P-P'')\, d^3p_1''\, d^3p_2''$$

$$\sigma \frac{V}{(2\pi)^4}$$

putting everything together

$$G_T = \frac{(2\pi)^3}{V} \quad i \langle P_1 P_2 | (T - T^+) | P_1 P_2 \rangle$$

recall $F = -(2\pi)^2 \mu\hbar \langle T \rangle$

so expressing this in terms of the scatter amplitude with $V = \frac{k}{\mu}$ gives

$$G_T = \frac{(2\pi)^3}{k/\mu} \quad i \frac{1}{-(2\pi)^2 \mu\hbar} (F - F^*)$$

$$= \frac{2\pi}{k\hbar} \underbrace{-i(F - F^*)}_{2\,\mathrm{Im}\,F(\bar{k}\bar{k})}$$

$\boxed{G_T = \frac{4\pi}{|k|\hbar} \text{ Imaginary part of forward scattering amplitude}}$

This is called the optical theorem. It is a consequenc of the unitarity of S.

The total cross section requires computing an integral – this is avoided by calculating F and looking at the forward part of F